Tesi di Dottorato

in

Statistica Metodologica

# Approach to Markovianity for random motions on hyperbolic spaces and time inhomogeneous jump processes

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# Contents

# Introduction

Markov processes are used to model a great variety of phenomena. Intuitively, a stochastic process is Markovian if the prediction of the future behaviour is at most determined by the present state only, while the whole past history is not relevant.

From the analytical point of view, each Markov process corresponds to a family of linear operators acting on a Banach space of functions. For time-homogeneous (resp. inhomogeneous) evolutions such a family forms a semigroup (resp. a propagator) of bounded operators. The main point in the construction of a Markov process is the definition of an operator which is called "generator", that gives rise to the corresponding evolutionary equations.

The protagonists of this thesis are the jump-type Markov processes: on the one hand the random flights on manifolds (where jumps occurr in the velocity), on the other Lévy processes and subordinators. These two topics respectively correspond to part I (chapters 1,2,3) and part II (chapters 4,5,6) of the present manuscript.

Part I is placed in a strand of literature that studies how a Markovian random flight in the euclidean space can be approximated by means of a sequence of non-Markovian processes. More precisely, the (not-Markovian) Lorentz Process, which is a motion among randomly distribuited obstacles, converges to a Markovian flight under a suitable scaling limit. The first attempt in this direction was performed by the physicist Giovanni Gallavotti and was developed in successive papers. In his study on the Lorentz Process and its Markovian approximation, Gallavotti was able to solve one of the biggest open problems of mathematical physics, namely the rigorous derivation of the Boltzmann equation. This is the equation governing the evolution of an interacting point-particle gas, namely a random flight with jumps of its velocity. In this thesis, I have studied the non-Euclidean counterpart of the Lorentz Process on a hyperbolic space, and the Boltzmann generator on the hyperbolic manifold has been obtained

Indeed, random motions in non-euclidean manifolds have been widely studied in recent years. The motivation for these studies ranges from the pure mathematical curiosity up to the justification given by modern physics, according to which the universe would not be ruled by Euclidean geometry. Such models often consist in the non-Euclidean counterpart of the most important stochastic processes already studied in the euclidean case. For example, hundreds of papers are devoted to the Brownian motion on the sphere and on hyperbolic spaces, as well as to random flights on Riemannian manifolds.

Surprisingly, no work has ever been written on the Lorentz process on non-Euclidean spaces, and this has prompted us to make our attempt, that is certainly the first of its kind ever to have been made.

Part II concerns Lévy processes and subordinators. In the literature, subordinators are used to the random time change. In particular, for a given process $X(t)$, the deterministic time $t$ can be substituted by a subordinator: this speeds up or slows down the process in a random way. The subordination of some Markov processes is the object of chapter 5. Explicit expressions for the state probabilities, for the intertimes between jumps and sojourn times are obtained in the case of birth, death and birth-death processes.

From a probabilistic point of view, the most technical part of the present manuscript is chapter 6, where a new class of markovian jump processes is investigated. These are called *non-homogeneous subordinators*, because they generalize the classical subordinators, in the sense that their increments are independent but non-stationary. Their time-dependent generators are determined and subordinated semigroups are investigated; a time-nonhomogeneous generalisation of the well known Phillips formula is proposed.

The plan of this manuscript is the following:

- The general theory of Markov processes (including the notions of semigroups, propagators and generators) is taken for granted.

- In Chapter 1 the Lorentz process in the Euclidean plane is described, and Gallavotti's results (together with its recent developements) are expounded.

- Chapter 2 is a self-consistent introduction on hyperbolic geometry.

- In Chapter 3, the Lorentz process on the hyperbolic half-plane is constructed. The hyperbolic version of Gallavotti's theorem is obtained and the Boltzmann Markovian generator is derived.

- Chapter 4 is an introduction to Lévy processes and Subordinators.

- Chapter 5 regards the composition of birth, death, and birth-death processes with standard subordinators
- Chapter 6 deals with non-homogeneous subordinators
- Fractional derivatives appear in Chapters 5 and 6 for some particular processes among the more general ones studied, but an introduction on fractional operators is voluntarily missing. Those who are not expert in fractional calculus can skip these parts without altering the understanding of the whole text.

Moreover, we stress that chapters 1, 2 and 4 are introductory and expose results that are known in the literature (although suitably reworked), while chapters 3, 5 and 6 contain original results, which are expounded in the following papers of which I am co-author:

- Population models at stochastic times, E.Orsingher, C. Ricciuti and B. Toaldo, *Advances in Applied Probability*, 2016.
- Motion among random obstacles on a hyperbolic space, E. Orsingher, C. Ricciuti and F. Sisti, *Journal of Statistical Physics*, 2016.
- Time-inhomogeneous jump processes and variable order operators, E. Orsingher, C. Ricciuti and B. Toaldo, *Potential Analysis*, 2016.

# Chapter 1

# Motion among random obstacles

## 1.1 Generalities

The subject of random obstacles aroused great interest in the last decades. Indeed random media are able to describe a great variety of phenomena, which go far beyond the behaviours displayed by periodic or constant media.

There are many applications in physics. For example, when an elementary particle moves through a material medium, it may be subject to the presence of randomly distribuited impurities, which act as either absorbing traps or reflective barriers. The same thing occurs when a ray of light passes through a transparent medium.

In many models, obstacles are assumed to be distribuited according to a poissonian field. A relevant attention has been given to Brownian motion among poissonian obstacles; on this point consult, for example, [68].

In this work, we are rather interested in the Lorentz model, describing a deterministic motion in presence of randomly distribuited obstacles. This model has a physical origin, as it was proposed by Lorentz to describe the motion of electrons inside a metallic material. Indeed, the classical model of electrical conduction states that the motion of the electrons is subject to several collisions with the metallic nuclei.

In the 1970's, thanks to the work of Gallavotti (see [26]), the Lorentz model was involved in what is perhaps one of the most important questions of mathematical physics: the rigorous derivation of the famous Boltzmann equation. Such an equation describes the macroscopic behavior of a gas (for a complete discussion see [20]), giving the time evolution of the mass density $f(q, v, t)$, namely the portion of particles located in $q$ with velocity $v$ at time $t$. A big problem of statistical physics was the inability to obtain the Boltzmann equation from the microscopic particle dynamics in a rigorous way, since it was necessary to make physical assumptions. On this point Gallavotti proved that the linear Boltzmann equation can be obtained rigorously from the microscopic dynamics in the case of the stochastic Lorentz gas by means of a scaling limit.

Leaving aside the dynamics of a gas, one can think in terms of stochastic processes: with this view, the mass density of the Lorentz gas is substituted by the probability density of a single particle moving among random obstacles. In this framework, we now show Gallavotti's results in more detail.

## 1.2 The Euclidean Lorentz Process

We now describe Gallavotti's results, which are reported in [26]. Suppose that static spherical obstacles are distributed in $\mathbb{R}^2$ according to a Poisson probability measure with intensity $\lambda$. More precisely, the probability to have $n$ obstacles centers in a region $S$ is given by

$$\Pr\{N(S) = n\} = e^{-\lambda|S|}\frac{(\lambda|S|)^n}{n!}$$

where $|S|$ is the area of $S$.

The particle is assumed to move along straight lines at constant velocity (which is assumed equal to 1 for the sake of simplicity) and to be specularly reflected by the obstacles. Therefore, a deterministic motion in a random environment arises. Thus the couple position-velocity at time $t$, denoted as $(Q(t), V(t))$, defines a stochastic process with values in $\mathbb{R}^2 \times S_1$, that we call Lorentz process (here $S_1$ denotes the unitary circle).

This process is clearly non-Markovian, because the trajectories remember the effect of previous collisions. The main result of Gallavotti's work is the proof of consistency of the so-called Boltzmann-Grad limit, in which the radius $r$ of each obstacle decreases to zero and the density $\lambda$ increases to infinity, so that the mean free path $(2\lambda r)^{-1}$ remains constant (few collisions regime). Under the Boltzmann-Grad asymptotics the one-dimensional probability density of the Lorentz process converges to that of a Markovian process, solving the following linear Boltzmann equation:

$$\frac{\partial}{\partial t}f(q,v,t) + v\cdot\nabla_q f(q,v,t) = -\sigma f(q,v,t) + \sigma\int_0^{2\pi} f(q,R_\beta,t)\sin\frac{\beta}{2}\frac{d\beta}{4} \tag{1.2.1}$$

where $R_\beta$ is the rotation of an angle $\beta$. Equation (1.2.1) is the generator equation of a markovian random flight. Such a process can be characterized as follows: a poissonian process with rate $\sigma$ governs the times in which changes of velocity in $S_1$ occurs; moreover, at such poissonian times, the velocity vector is rotated by an angle $\beta \in [0, 2\pi]$ with distribution density $\frac{1}{4} \sin \frac{\beta}{2}$.

The meaning of this result is quite intutive: the probability of having recollisions (i.e. more than one collision with a given obstacle) for the particle vanishes in Boltzmann-Grad asymptotics and a markovian dynamics arises.

Gallavotti's model was firstly improved by Spohn [67] and Boldrighini et al. [13], and was further developed in successive papers. In [21], Desvillettes and Ricci studied a variant of the model in which the obstacles are totally absorbing and an external force field is present; in this case the Boltzmann-Grad limit does not lead to a Markovian process, unless a random motion of the obstacles is assumed (with Gaussian distribution of velocities). Gallavotti's work has also inspired the approach of Basile, Nota and Pulvirenti [7] where the context is slightly different, especially because the obstacles consist in circular potential barriers instead of hard spheres. However, the authors follow the same steps of Gallavotti's proof and, by suitably scaling the Poissonian density and the potential intensity, they obtain a Markovian approximation which is governed by a linear Landau equation

$$\frac{\partial}{\partial t} f(q, v, t) + v \cdot \nabla_q f(q, v, t) = B \Delta_{|v|} f(q, v, t) \tag{1.2.2}$$

This is the generator equation of a Brownian motion in $S_1$. Therefore, unlike what happens with the Boltzmann-Grad limit, where the mean free path is kept constant and the velocity changes are regulated by a Poisson process, here the qualitative behavior is completely different, since there is an infinite number of speed changes in each time interval.

## 1.3 Random obstacles in non-Euclidean spaces

Random motions in non-Euclidean spaces have attracted the interest of many mathematicians in the last decades. The main reason of this interest lies in the fact that modern physics states that the universe is governed by non euclidean geometries.

In particular, random models in hyperbolic spaces have a prominent role in the literature. Most of the papers are devoted to the hyperbolic Brownian motion, but recently random motions at finite velocity have also been studied. The main references on this topic are [17; 18; 19; 52]). The scheme underlying these works is the following: the particle moves along geodesics (namely, the curves that play the same role of straight lines in euclidean spaces) and changes direction at Poisson times according to some transition probability law.

In this thesis, we introduce for the first time the notion of random obstacles in non-euclidean spaces. We consider the motion of a particle along the geodesic lines of the Poincaré half-plane, which is one of the most important models of hyperbolic geometry. The particle is specularly reflected when hitting randomly distribuited obstacles that are supposed to be motionless. This is the hyperbolic version of the Lorentz Process studied by Gallavotti in the Euclidean context. We then analyse the limit in which the density of the obstacles increases to infinity and the size of each obstacle vanishes: under a suitable scaling, we prove that our process converges to some Markovian process, namely to a random flight on the hyperbolic manifold, which is governed by a Boltzmann-type generator.

Before describing in detail our model and results we need to introduce the main concepts of hyperbolic geometry, with particular reference to the Poincaré half-plane model, and this is the role of the following chapter.

# Chapter 2

# Elements of hyperbolic geometry

## 2.1 Introduction

In this chapter we give some basic notions of hyperbolic geometry, with a particular attention to the Poincaré half plane. We obviously do not claim to be exhaustive: for a detailed discussion refer to the books of riemmanian and hyperbolic geometry (see, for example, [14]).

Hyperbolic geometry had among its founders distinguished mathematicians such as Saccheri, Bolyai and Lobachevsky. It was born first of all to solve one of the key issues of Euclidean geometry, concerning the fifth Euclid's postulate. This postulate states that "given a line $r$ and a point $P$ not contained in $r$, there exists a unique line parallel to $r$ (i.e. not intersecting $r$) and containing $P$". Indeed it was uncertain whether that statement was really a postulate, or if it could be deduced from the others. Hyperbolic geometry originated just trying to deny the fifth postulate: there came out an elegant and perfectly consistent theory with its properties and definitions. This showed once and for all the independence of the fifth postulate from the other four.

The actual consistency of hyperbolic geometry is guaranteed by the existence of some models, which are substantially equivalent to each other. These include the Poincaré half-plane, the Poincaré disk, the Klein disk and the hyperboloid.

## 2.2 The Poincaré half-plane.

The Poincaré half-plane is the region

$$H_2 = \{(x, y) \in \mathbb{R}^2, y > 0\} \tag{2.2.1}$$

endowed with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{2.2.2}$$

The geometry in $H_2$ follows from (2.2.2). As a first step, we determine the geodesics, i.e. the curves having the role of lines in the Poincare half-plane. From (2.2.2) it follows that the role of geodesic curves is played by all the Euclidean half-circles with the center on the x-axis and by all the Euclidean lines parallel to the y-axis, as shown in the following theorem.

**Theorem 1.** *Among all the curves connecting two given points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, with $x_1 \neq x_2$, the shortest one is an arc of an euclidean half-circle with its center on the $x$-axis. Moreover, among all the curves connecting two given points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, with $x_1 = x_2$, the shortest one is the vertical line connecting them.*

*Proof.* Let $\gamma$ be a curve in the hyperbolic half-plane:

$$\gamma = \{(x(t), y(t)) : \quad t_1 \leq t \leq t_2\} \tag{2.2.3}$$

The hyperbolic length of $\gamma$ is defined as:

$$\mathcal{L}(\gamma) = \int_{t_1}^{t_2} \frac{\sqrt{x'^2(t) + y'^2(t)}}{y(t)} dt \tag{2.2.4}$$

If $\gamma$ is the plot of a function $y = y(x)$, the previous formula reduces to the functional

$$\mathcal{L}[y] = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2(x)}}{y(x)} dx \tag{2.2.5}$$

If $y = y(x)$ is the geodesic line, any other curve joining $P_1$ and $P_2$ can be written as

$$w(x) = y(x) + \epsilon h(x) \qquad \epsilon \geq 0$$

for a given function $h$ satisfying

$$h(x_1) = h(x_2) = 0 \tag{2.2.6}$$

. By fixing $h$, the functional $\mathcal{L}[w] = \mathcal{L}[y + \epsilon h]$ simply reduces to the function

$$l(\epsilon) = \int_{x_1}^{x_2} \frac{\sqrt{1 + (y' + \epsilon h')^2}}{y + \epsilon h} dx \tag{2.2.7}$$

As we assumed that $y$ is the shortest curve, $l(\epsilon)$ has a minimum for $\epsilon = 0$:

$$\frac{\partial l}{\partial \epsilon}|_{\epsilon=0} = 0 \tag{2.2.8}$$

$$\int_{x_1}^{x_2} dx \left( \frac{\partial}{\partial \epsilon} \frac{\sqrt{1+(y'+\epsilon h')^2}}{y+\epsilon h} \right)_{\epsilon=0} = 0$$

$$\int_{x_1}^{x_2} dx \left( \frac{\frac{2(y'+\epsilon h')h'}{2\sqrt{1+(y'+\epsilon h')^2}}(y+\epsilon h) - h\sqrt{1+(y'+\epsilon h')^2}}{(y+\epsilon h)^2} \right)_{\epsilon=0} = 0$$

$$\int_{x_1}^{x_2} dx \left( \frac{y'}{y\sqrt{1+y'^2}} h' - \frac{\sqrt{1+y'^2}}{y^2} h \right) = 0$$

We integrate by parts the first term, reminding condition (2.3.6), obtaining

$$\int_{x_1}^{x_2} \Big( -\frac{d}{dx} \frac{y'}{y\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}}{y^2} \Big) h(x) dx = 0$$

But the last condition must be true for any fixed function $h$, hence

$$-\frac{d}{dx} \frac{y'}{y\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}}{y^2} = 0 \tag{2.2.9}$$

which is the differential equation giving the geodesic line $y = y(x)$.
After some calculations, eq. (2.2.9) reduces to

$$y''y + (y')^2 + 1 = 0$$

To find the explicit solution, it is sufficient to write

$$\frac{d}{dx}(yy') + 1 = 0 \Longrightarrow yy' = -x + c$$

By separation of variables, we have

$$\int_{y_0}^{y} Y dY = \int_{x_0}^{x} (-X + c) dX$$

which gives the equation of a circle having the centre on the $x-$ axis:

$$x^2 + y^2 + c_1 x + c_2 = 0$$

for suitable constants $c_1$ and $c_2$ determined by the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. The first part of the theorem is thus proved.

For any curve $\gamma$ connecting $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, we have

$$\mathcal{L}(\gamma) = \int_{t_1}^{t_2} \frac{\sqrt{x'^2(t) + y'^2(t)}}{y(t)} dt \geq \int_{t_1}^{t_2} \frac{|y'(t)|}{y(t)} dt$$

and lower bound is just the lenght of the vertical segment connecting the two points.

$\square$

We now observe that the celebrated fifth Euclidean axiom fails to be true in $H_2$. Indeed, given a geodetic curve $\gamma$ and a point $P$ outside $\gamma$, there is not a unique geodesic line containing $P$ and parallel to $\gamma$.

## 2.3 Digression on geodesics

This section presents an optical interpretation on the metric of the Poincaré half-plane. It's worth noting that this section can be skipped without affecting the understanding of the subsequent discussion.

The determination of the geodesics is fundamental to study the propagation of light in a non- homogeneous medium.

The time required to travel the path $\gamma$ is

$$T(\gamma) = \int_\gamma dt = \int_\gamma \frac{\sqrt{dx^2 + dy^2}}{v} \tag{2.3.1}$$

where $v$ is the light velocity.
If the light moves in a medium with refracting index $n$, we have

$$v = \frac{c}{n} \tag{2.3.2}$$

where $c$ is the light speed in the vacuum . Clearly, $n$ is constant in a homogeneous medium, while $n = n(x, y)$ in a non-homogeneous one.
According to the Fermat principle, if a ray of light moves from $P_1$ to $P_2$, it chooses the path that takes the least time.
So the problem of determining geodetics consists in finding the minimum of $T(\gamma)$ among all the possible curves $\gamma$ connecting $P_1$ and $P_2$.
In particular, the Poincare' half plane can be seen as a non-homogeneous medium with $n(x, y) = \frac{1}{y}$. So, we have to minimize

$$T(\gamma) = \int_\gamma dt = \frac{1}{c}\int_\gamma \frac{\sqrt{dx^2 + dy^2}}{y} \tag{2.3.3}$$

Such physical problem is equivalent to defining a metric

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}$$

and minimizing the lenght of a curve, as explained in the previous section.
An imediate generalisation is to consider a refracting index $n(x, y) = \frac{1}{y^\alpha}$, so the time to minimize is:

$$T(\gamma) = \int_\gamma dt = \frac{1}{c}\int_\gamma \frac{\sqrt{dx^2 + dy^2}}{y^\alpha} \tag{2.3.4}$$

We now show that the solution to this problem is a curve $y = y(x)$ solving the following differential equation:

$$y''y + \alpha y'^2 + \alpha = 0 \tag{2.3.5}$$

We remark that the last equation contains two special cases: if $\alpha = 0$, we have $y'' = 0$ which furnishes the classical euclidean lines; if $\alpha = 1$ we obtain the hyperbolic lines, as already studied in the previous section.

The proof is analogous to the one in the previous section. We repeat all the passages, in order to make this section self-consistent.

If $y = y(x)$ is the geodesic line, any other curve joining $P_1$ and $P_2$ can be written as

$$w(x) = y(x) + \epsilon h(x) \qquad \epsilon \geq 0$$

for a given function $h$ satisfying

$$h(x_1) = h(x_2) = 0 \tag{2.3.6}$$

By fixing $h$, the functional $\mathcal{L}[w] = \mathcal{L}[y + \epsilon h]$ simply reduces to the function

$$l(\epsilon) = \int_{x_1}^{x_2} \frac{\sqrt{1 + (y' + \epsilon h'^2}}{(y + \epsilon h)^\alpha} dx \tag{2.3.7}$$

As we assumed that $y$ is the shortest curve, $l(\epsilon)$ has a minimum for $\epsilon = 0$:

$$\frac{\partial l}{\partial \epsilon}|_{\epsilon=0} = 0 \tag{2.3.8}$$

So, we have:

$$\int_{x_1}^{x_2} dx \left( \frac{\partial}{\partial \epsilon} \frac{\sqrt{1 + (y' + \epsilon h')^2}}{(y + \epsilon h)^\alpha} \right)_{\epsilon=0} = 0$$

$$\int_{x_1}^{x_2} dx \left( \frac{\frac{2(y'+\epsilon h')h'}{2\sqrt{1+(y'+\epsilon h')^2}}(y + \epsilon h)^\alpha - \alpha(y + \epsilon h)^{\alpha-1} h\sqrt{1 + (y' + \epsilon h')^2}}{(y + \epsilon h)^{2\alpha}} \right)_{\epsilon=0} = 0$$

$$\int_{x_1}^{x_2} dx \left( \frac{y'}{y^\alpha \sqrt{1 + y'^2}} h' - \frac{\alpha\sqrt{1 + y'^2}}{y^{\alpha+1}} h \right) = 0$$

We integrate by parts the first term, obtaining

$$\int_{x_1}^{x_2} \Big( -\frac{d}{dx} \frac{y'}{y^\alpha \sqrt{1 + y'^2}} - \frac{\alpha\sqrt{1 + y'^2}}{y^{\alpha+1}} \Big) h(x) dx = 0$$

But the last condition must be true for any $h$, so that

$$-\frac{d}{dx} \frac{y'}{y^\alpha \sqrt{1 + y'^2}} - \frac{\alpha\sqrt{1 + y'^2}}{y^{\alpha+1}} = 0$$

After some calculation, it reduces to

$$y''y + \alpha y'^2 + \alpha = 0$$

and the proof is complete.

To solve such equation, we first divide both members for $y\,(y'^2 + 1)$ and multiply them for $y'$:

$$\frac{y''y'}{y'^2+1} + \alpha\frac{y'}{y} = 0$$

Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dx}log(y'^2+1) + \alpha\frac{d}{dx}logy = 0\\ \frac{d}{dx}log[y^\alpha(y'^2+1)^{\frac{1}{2}}] = 0\\ y^\alpha(y'^2+1)^{\frac{1}{2}} = e^c\end{aligned}$$

Thus, we obtain a first order autonomous equation:

$$y' = \sqrt{k^2y^{-2\alpha} - 1}$$

where $k = e^c$. By separation of variables, we have

$$\int_{y_0}^{y} \frac{Y^\alpha}{\sqrt{k^2 - Y^{2\alpha}}}dY = \int_{x_0}^{x} dX \tag{2.3.9}$$

Using the substitution $Y^\alpha = k\cos\phi$, we can write

$$\int \frac{Y^\alpha}{\sqrt{k^2 - Y^{2\alpha}}}dY = -\frac{k^{\frac{1}{\alpha}}}{\alpha}\int(\cos\phi)^{\frac{1}{\alpha}}d\phi$$

The last integral can be easily solved in the case $\alpha = \frac{1}{3}$:

$$\begin{aligned}-3k^3\int(\cos\phi)^3 d\phi &= -3k^3\int\left(\cos\phi - \cos\phi\sin^2\phi\right)d\phi = -3k^3\sin\phi + k^3\sin^3\phi\\ &= -3k^2\sqrt{k^2 - Y^{\frac{2}{3}}} + \sqrt{\left(k^2 - Y^{\frac{2}{3}}\right)^3}\end{aligned}$$

where, in the last equality, we inverted $\cos\phi = \frac{Y^{\frac{1}{3}}}{k}$ to obtain $\sin\phi = \sqrt{1-\cos^2\phi} = \frac{1}{k}\sqrt{k^2 - Y^{\frac{2}{3}}}$.

Solving the integrals in eq (2.3.9), we obtain

$$-3k^2\sqrt{k^2 - y^{\frac{2}{3}}} + \sqrt{\left(k^2 - y^{\frac{2}{3}}\right)^3} = x + c \tag{2.3.10}$$

for a suitable constant $c$. By squaring and simplifying, we easily have

$$-y^2 - 3k^2y^{\frac{4}{3}} = x^2 + 2cx + d$$

which is the solution $y = y(x)$ in an implicit form.

## 2.4 Isometries in $H_2$

In solving problems of Euclidean geometry, we are used to move or rotate the Cartesian axes in order to observe objects of interest in the simplest way possible. This procedure is possible because the Euclidean space is invariant under rotations and translations. In more technical terms, translations and rotations are isometries in Euclidean space, namely operations that preserve the measure of lengths, areas and angles.

Even in the hyperbolic plane it is often useful to look at objects from a convenient point of view. To do this it is helpful to know what are the existing isometries. This will be very useful both in determining the geometric properties, as we shall see in the next sections, and in solving dynamical problems, as we will show in the next chapter.

All isometries of $H_2$ (i.e operations preserving lengths, areas and angles) are given by the so-called Möbius group of tranformations. The Möbius group can be easily defined by using an equivalent definition of Poincaré half-plane. Indeed $H_2$ can be defined in the complex domain as $H_2 = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ where the measure of the infinitesimal arc length is given by $\frac{|dz|}{Imz}$. In this setting the function $\mathcal{M} : \mathbb{C} \to \mathbb{C}$ is a Möbius transform if and only if it has the form

$$\mathcal{M}(z) = \frac{az+b}{cz+d} \qquad \text{with } a, b, c, d \in \mathbb{R};\ ad - bc = 1 \tag{2.4.1}$$

Among (2.4.1) there are the horizontal translations $(x, y) \to (x+c, y)$ and the homotheties $(x, y) \to (\lambda x, \lambda y)$. We will also do extensive use of the Möbius tranformation described in ([14], lemma 2.6), which acts by changing a given geodesic line into a vertical geodesic line. Such an isometry simplifies enormously dynamical problems: it is clear that it is much easier to deal with trajectories represented by euclidean straight lines, compared to trajectories represented by curves.

## 2.5 Hyperbolic distance

**Proposition 1.** *Let $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ be two points in $H_2$. Then the hyperbolic distance $\eta$ between $P$ and $Q$ is given by*

$$\cosh\eta = \frac{(x_P - x_Q)^2 + y_P^2 + y_Q^2}{2y_P y_Q} \tag{2.5.1}$$

*Proof.* We first prove that the distance between the origin $O = (0, 1)$ and a generic point $(x, y)$ is given by

$$\cosh \eta = \frac{x^2 + y^2 + 1}{2y}. \tag{2.5.2}$$

Consider the geodesic line passing in $O = (0, 1)$ with tangent vector $(\cos \alpha, \sin \alpha)$. It corresponds to an euclidean half-circle with radius $\frac{1}{\cos \alpha}$ and center $(\tan \alpha, 0)$. The generic point on this geodesic can be written in parametric form as

$$\begin{cases} x(\theta) = \tan \alpha + \frac{1}{\cos \alpha} \cos \theta \\ y(\theta) = \frac{1}{\cos \alpha} \sin \theta \end{cases} \qquad \theta \in [0, \pi) \tag{2.5.3}$$

while the point $(x, y)$ is such that

$$\tan \alpha = \frac{x^2 + y^2 - 1}{2x}. \tag{2.5.4}$$

Then the geodesic distance between $(0, 1)$ and $(x, y)$ can be computed as

$$\eta = \int_O^P \frac{\sqrt{dx^2 + dy^2}}{y} = \int_{\arcsin(\cos \alpha)}^{\arcsin(y \cos \alpha)} \frac{1}{\sin \theta} d\theta = \log \frac{\tan \frac{\arcsin\{y \cos \alpha\}}{2}}{\tan \frac{\arcsin\{\cos \alpha\}}{2}} = \tag{2.5.5}$$

$$= \log \frac{\sqrt{\frac{1-\cos(\arcsin(y \cos \alpha))}{1+\cos(\arcsin(y \cos \alpha))}}}{\sqrt{\frac{1-\cos(\arcsin(\cos \alpha))}{1+\cos(\arcsin(\cos \alpha))}}} = \log \frac{\sqrt{\frac{1-\sqrt{1-y^2 \cos^2 \alpha}}{1+\sqrt{1-y^2 \cos^2 \alpha}}}}{\sqrt{\frac{1-\sin \alpha}{1+\sin \alpha}}} \tag{2.5.6}$$

$$= \log \left( \sqrt{\frac{1 - \sqrt{1 - y^2 \cos^2 \alpha}}{1 + \sqrt{1 - y^2 \cos^2 \alpha}}} \sqrt{\frac{1 + \sin \alpha}{1 - \sin \alpha}} \right) \tag{2.5.7}$$

Then, after some calculations,

$$e^\eta + e^{-\eta} = \frac{2(1 - \sin \alpha \sqrt{1 - y^2 \cos^2 \alpha})}{y \cos^2 \alpha} \tag{2.5.8}$$

which can be written as

$$\cosh \eta = \frac{\sin^2 \alpha + \cos^2 \alpha - \sin \alpha \sqrt{\cos^2 \alpha + \sin^2 \alpha - y^2 \cos^2 \alpha}}{y \cos^2 \alpha} \tag{2.5.9}$$

$$= \frac{1 + \tan^2 \alpha - \tan \alpha \sqrt{1 + \tan^2 \alpha - y^2}}{y} \tag{2.5.10}$$

Then it is sufficient to use (2.5.4) and to recognize the square trinomial under the root:

$$\sqrt{\frac{(x^2 - y^2 + 1)^2}{4x^2}} = -\frac{x^2 - y^2 + 1}{2x} \tag{2.5.11}$$

and (2.5.2) is obtained. We finally take into account that the hyperbolic plane is invariant under homothety and horiziontal translation. Thus, the distance between

$(x_P, y_P)$ and $(x_Q, y_Q)$ is equal to the distance between the two points $(0,1)$ and $(\frac{x_P - x_Q}{y_Q}, \frac{y_P}{y_Q})$, that is

$$\cosh\eta = \frac{(\frac{x_P - x_Q}{y_Q})^2 + (\frac{y_P}{y_Q})^2 + 1}{2\frac{y_P}{y_Q}} = \frac{(x_P - x_Q)^2 + y_P^2 + y_Q^2}{2y_P y_Q} \tag{2.5.12}$$

and the proof is complete. □

An hyperbolic circumference of center $(x_C, y_C)$ and radius $\eta$ is defined as the set of all points of $H_2$ having hyperbolic distance $\eta$ from $(x_C, y_C)$, From (2.5.1) we obtain the equation of the hyperbolic circumference of radius $\eta$ and center $(x_c, y_c)$:

$$B_\eta(C) : \{(x,y) \, : \, (x - x_c)^2 + y^2 - 2yy_c\cosh\eta + y_c^2 = 0\} \tag{2.5.13}$$

corresponding to an Euclidean circumference of radius $y_c \sinh\eta$ and center $(x_c, y_c\cosh\eta)$.

## 2.6 Hyperbolic area

The infinitesimal hyperbolic area in $H_2$ is defined as

$$dA = \frac{dxdy}{y^2} \tag{2.6.1}$$

whence the hyperbolic area of a region $S \subset H_2$ is defined as

$$||S|| = \int_{\mathcal{S}} \frac{dxdy}{y^2} \tag{2.6.2}$$

In some cases it is convenient to write such an integral in hyperbolic coordinates. A point $P = (x,y) \in H_2$ can be expressed by the hyperbolic coordinates $(\eta, \alpha)$, where $\eta$ is the distance between $P$ and the origin $O = (0,1)$, while $(\cos\alpha, \sin\alpha)$ is the tangent vector at $O$ to the geodesic line connecting $O$ and $P$. The change of variable is expressed by

$$x = \frac{\sinh\eta\cos\alpha}{\cosh\eta - \sin\alpha\sinh\eta} \tag{2.6.3}$$

$$y = \frac{1}{\cosh\eta - \sin\alpha\sinh\eta} \tag{2.6.4}$$

as will be proved in a more general context in the appendix of the following chapter.

The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial x}{\partial\eta} & \frac{\partial x}{\partial\alpha} \\ \\ \frac{\partial y}{\partial\eta} & \frac{\partial y}{\partial\alpha} \end{pmatrix} = \begin{pmatrix} \frac{\cos\alpha}{(\cosh\eta - \sin\alpha\sinh\eta)^2} & \frac{sinh^2\eta - \sin\alpha\sinh\eta\cosh\eta}{(\cosh\eta - \sin\alpha\sinh\eta)^2} \\ \\ \frac{\sin\alpha\cosh\eta - \sinh\eta}{(\cosh\eta - \sin\alpha\sinh\eta)^2} & \frac{\cos\alpha\sinh\eta}{(\cosh\eta - \sin\alpha\sinh\eta)^2} \end{pmatrix} \tag{2.6.5}$$

with

$$detJ = \frac{\sinh \eta}{(\cosh \eta - \sin \alpha \sinh \eta)^2}. \tag{2.6.6}$$

So, being $dxdy = |detJ| d\eta d\alpha$, the above integral in hyperbolic coordinates imediately follows:

$$||S|| = \int_{S'} \sinh \eta d\eta d\alpha \tag{2.6.7}$$

Such a expression can be used to compute the area of regions having a radial simmetry. For example, let's consider a hyperbolic ball of radius $r$, which is defined as

$$B = \{(\eta, \alpha) : \quad \eta \leq r, \quad 0 \leq \alpha \leq 2\pi\}$$

The area can be computed as

$$||B|| = \int_0^{2\pi} d\alpha \int_0^r d\eta \sinh \eta = 4\pi \sinh^2 \left(\frac{r}{2}\right) \tag{2.6.8}$$

## 2.7 Other models of hyperbolic space.

Another famous model of hyperbolic geometry is the Poincaré disk $D_2$, defined as the region $D_2 = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 1\}$ endowed with the metric

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}. \tag{2.7.1}$$

The Poincaré half-plane and the Poincaré disk are isomorphic. The isomorphism is given by the Cayley transform: a point $(x, y) \in H_2$ is mapped into the point $(u, v) \in D_2$ with coordinates

$$u = \frac{2x}{x^2 + (y+1)^2} \qquad v = \frac{x^2 + y^2 - 1}{x^2 + (y+1)^2}.$$

while the inverse mapping reads

$$x = \frac{2u}{u^2 + (1-v)^2} \qquad y = \frac{1 - (u^2 + v^2)}{u^2 + (1-v)^2}.$$

Then it is easy to show that (2.7.1) is a direct consequence of (2.2.2). Indeed it is sufficient to compute the partial derivatives:

$$\frac{\partial x}{\partial u} = \frac{2(1-v)^2 - 2u^2}{(u^2 + (1-v)^2)^2} \qquad \frac{\partial x}{\partial v} = \frac{4u(1-v)}{(u^2 + (1-v)^2)^2}$$
$$\frac{\partial y}{\partial u} = \frac{-4u(1-v)}{(u^2 + (1-v)^2)^2} \qquad \frac{\partial x}{\partial v} = \frac{2(1-v)^2 - 2u^2}{(u^2 + (1-v)^2)^2}$$

and consider the differentials

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$
$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

whence (2.7.1) immediately follows by a simple substitution.

In particular, the image of the $x$-axis of $H_2$ is the border of $D_2$. Moreover, the Cayley transform is conformal, namely it preserves angles and maps geodesic lines into geodesic lines, hyperbolic circles into hyperbolic circles. In particular, if a geodesic line in $H_2$ is represented by an euclidean half circles with center $(x_0, 0)$ and radius $r$, its image is given by an arc of circumference which is orthogonal to the border od $D_2$, having center at

$$\left(\frac{2x_0}{x_0^2 - r^2 + 1}, \frac{x_0^2 - r^2 - 1}{x_0^2 - r^2 + 1}\right)$$

and radius $R$ such that $R^2 = (\frac{2r}{x_0^2 - r^2 + 1})^2$.

We underline that the disc $D_2$ and the half-plane $H_2$ share an important feature, in that they both are conformal models of hyperbolic geometry: hyperbolic angles between incident geodesic lines correspond to angles measured by an euclidean obseverver. Moreover hyperbolic circles correspond to euclidean circles.

Another model of hyperbolic geometry is the Kein disk model. In the Klein disk the role of geodesic lines is given by euclidean chords. However, the greater simplicity given by the shape of geodesics is compensated by the fact that the angles are distorted, i.e. they do not correspond to euclidean angles. Even the circles are distorted, as the appear elliptical to an euclidean observer.

# Chapter 3

# The hyperbolic Lorentz process

## 3.1 Introduction

In this chapter we treat the Lorentz process in the Poincaré half plane. Such a model is completely described by three ingredients: the free geodesic flow, the particle-obstacle interaction and the obstacles spatial distribution. We here give a brief description of the motion, further mathematical details will be given in the following sections.

The free particle moves along geodesic lines and is reflected by the scatterers. From (2.2.2) the intensity of the hyperbolic velocity is defined as

$$v_{hyp}(t) = \frac{ds}{dt} = \frac{1}{y}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{v_{euc}}{y}, \tag{3.1.1}$$

We here assume that the particle moves at constant hyperbolic velocity $v_{hyp}(t) = c$. This means that the hyperbolic distance run by the particle in a time $\Delta t$ is given by $c\Delta t$. Therefore, an Euclidean observer sees the particle moving with a position-dependent velocity equal to $v_{euc} = c\,y$. Without loss of generality, one can assume $c = 1$.

We introduce the notion of Poissonian distribution of obstacles in the Poincaré hyperbolic half-plane. The obstacles are hyperbolic balls of radius $r$, whose centers are distributed according to a spatial Poisson process which is homogeneous in the sense of the measure (2.6.1), i.e. the mean number of obstacle centers per unit hyperbolic area (denoted as $\lambda$) is uniform in $H_2$. This means that the obstacles are identical and homogeneously distributed in respect to the hyperbolic metric, yet to an Euclidean observer they appear to be smaller and denser when approaching to the $x$-axis (see figure 3.4).

The main result of this study is the analysis of a Boltzmann-Grad-type limit in which the hyperbolic radius $r$ of each obstacle decreases to zero and the density $\lambda$ in the hyperbolic setting diverges to infinity, so that the mean free path $(2\lambda \sinh r)^{-1}$ remains constant. Under this limit, the density of our process converges to the density of some Markovian random flight. Moreover, we prove that this limit random motion is similar to the process analysed by M.Pinsky [63], who generalized the well-known Euclidean isotropic transport process to the case of an arbitrary Riemannian manifold.

## 3.2 Random obstacles in the Poincaré half-plane.

### 3.2.1 Poisson random fields in $H_2$

Assume that a countable set $\Pi$ of points is randomly distributed on the Poincaré half-plane $H_2$ with rate $\lambda(x, y)$. We say that $\Pi$ is a Poisson random field in $H_2$ if:

- For any appropriate set $\mathcal{S} \subset H_2$, the random variable $N(\mathcal{S})$, namely the cardinality of $\Pi \cap \mathcal{S}$, has the following distribution:

$$\Pr(N(\mathcal{S}) = k) = e^{-\Lambda(\mathcal{S})} \frac{(\Lambda(\mathcal{S}))^k}{k!} \tag{3.2.1}$$

  with

$$\Lambda(\mathcal{S}) = \int_{\mathcal{S}} \lambda(x, y) \frac{dxdy}{y^2}. \tag{3.2.2}$$

- For any couple of disjoint regions $\mathcal{S}_1$ and $\mathcal{S}_2$, the random variables $N(\mathcal{S}_1)$ and $N(\mathcal{S}_2)$ are stochastically independent.

We here restrict our attention to the case where the rate $\lambda$ is constant (homogeneous hyperbolic Poisson field). Thus, the number of points inside any set $\mathcal{S} \subset H_2$ has Poisson distribution with parameter $\lambda|\mathcal{S}|$, where

$$|\mathcal{S}| = \int_{\mathcal{S}} \frac{dxdy}{y^2} \tag{3.2.3}$$

is the hyperbolic area of $\mathcal{S}$.

Therefore the probability to have exactly $n$ points in a region $\mathcal{S}$ and to find them inside the hyperbolic elements $dc_1, dc_2, ...dc_n$ around $c_1, c_2, ...c_n$ is given by

$$\Pr\{P_1 \in dc_1 \dots P_n \in dc_n, N(t) = n\} = \lambda^n e^{-\lambda|\mathcal{S}|} dc_1....dc_n \tag{3.2.4}$$

where

$$dc_j = \frac{dx_j dy_j}{y_j^2}. \tag{3.2.5}$$

It is important to observe that the homogeneous Poisson random field only depends on the measure of areas and therefore it is invariant under the group of isometries of $H_2$ expressed in (2.4.1).

To have a more complete description of the homogeneous hyperbolic Poisson field, we treat briefly the distributions of the nearest neighbours points. Let us fix a point $O \in H_2$ and denote by $T_k$ the hyperbolic distance between $O$ and the $k^{th}$ nearest point of $\Pi$.

Denoting by $B_\eta$ the hyperbolic ball of radius $\eta$ and by $dB_\eta$ the infinitesimal anulus of radii $\eta$ and $\eta + d\eta$, we have

$$\begin{aligned}
\Pr\{T_k \in d\eta\} &= \Pr\{N(B_\eta) = k-1\} \Pr\{N(dB_\eta) = 1\} \\
&= e^{-\lambda|B_\eta|} \frac{(\lambda|B_\eta|)^{k-1}}{(k-1)!} \lambda|dB_\eta| \qquad k \geq 1, \eta > 0
\end{aligned} \tag{3.2.6}$$

Since $|B_\eta| = 4\pi \sinh^2 \frac{\eta}{2}$, the anulus $dB_\eta$ has measure $2\pi \sinh \eta d\eta$ and thus

$$\Pr(T_k \in d\eta) = e^{-4\pi\lambda \sinh^2 \frac{\eta}{2}} \frac{\left(4\pi\lambda \sinh^2 \frac{\eta}{2}\right)^{k-1}}{(k-1)!} 2\pi\lambda \sinh \eta d\eta \tag{3.2.7}$$

In particular, the distribution for the nearest neighbour $T_1$ reads

$$\Pr(T_1 \in d\eta) = e^{-4\pi\lambda \sinh^2 \frac{\eta}{2}} 2\pi\lambda \sinh \eta d\eta \tag{3.2.8}$$

with expectation

$$\mathbb{E}(T_1) = e^{2\pi\lambda} K_0(2\pi\lambda), \tag{3.2.9}$$

where

$$K_0(z) = \int_0^\infty e^{-z\cosh t} dt \tag{3.2.10}$$

is the modified Bessel function. Formula (3.2.8) is the hyperbolic counterpart of the well known Rayleigh distribution, which describes the distance $T_1^e$ of the nearest neighbour point in the case of a Poissonian random field in the Euclidean plane:

$$\Pr(T_1^e \in dr) = e^{-\lambda\pi r^2} 2\pi\lambda r dr \tag{3.2.11}$$

with mean value

$$E\left(T_1^e\right) = \frac{1}{2\sqrt{\lambda}}. \tag{3.2.12}$$

By means of the asymptotic formula for the modified Bessel function, expression (3.2.9) reduces to (3.2.12) for large values of $\lambda$, namely when the expected distance between Poissonian points decreases and an Euclidean description works well.

### 3.2.2 Poissonian obstacles

We now introduce the notion of the Poissonian distribution of obstacles into the hyperbolic half-plane $H_2$, and we distinguish between hard and soft obstacles, which is a common practice for motions in Euclidean spaces.

We are inspired by [53], where the author studies Poissonian soft obstacles in a particular non Euclidean manifold: the surface of a sphere.

Let us consider $\Pi$ to be a homogeneous hyperbolic Poisson field in $H_2$ with constant intensity $\lambda$ and let us assume that each point $P \in \Pi$ produces a potential around itself, whose intensity $\phi$ is a function of the geodesic distance from $P$. It is assumed that $\phi$ is compactly supported, namely $\phi(\eta) = 0$ for $\eta > r$. The hyperbolic ball of center $P$ and radius $r$, where the function $\phi$ is non-null, is known as a soft obstacle. When a particle hits a soft obstacle, it is subject to an interaction described by $\phi$. Of course, the obstacles may overlap, which occurs whenever the geodesic distance between two Poissonian points is less than $2r$. Therefore, at a certain point $Q$, the superposition of the action due to the points $P_1 ... P_N$ located in a hyperbolic ball $B_r(Q)$ defines a new random field

$$V(Q) = \sum_{j=1}^{N} \phi\big(d_h(P_j Q)\big) \tag{3.2.13}$$

where $N$ has Poisson distribution with parameter $\lambda|B_r(Q)|$ and $d_h\big(P_j Q\big)$ is the geodesic distance between $P_j$ and $Q$. Two facts play fundamental roles. The first is that the random field (3.2.13) is homogeneous, meaning that the distribution of $V(Q)$ does not depend on $Q$. The second is that (3.2.13) is isotropic, namely the covariance between $V(Q)$ and $V(Q')$ only depends on the geodetic distance between $Q$ and $Q'$. For the sake of brevity we omit a complete proof of these facts which can easily be obtained by following the same steps as in [53].

Hard obstacles are hyperbolic disks of radius $r$ centered at the points of a Poisson random field in $H_2$. In many models of random motions, hard obstacles represent totally absorbing traps with random locations. In other models, they act as totally reflecting barriers and can be deemed to be the limiting case of soft obstacles where the following intensity function is considered:

$$\phi(\eta) = \begin{cases} \infty & \eta \leq r \\ 0 & \eta > r \end{cases}$$

Thus, in what follows, we consider a system of hard obstacles, whose centers are distributed according to a homogeneous hyperbolic Poisson field of constant intensity $\lambda$. A configuration of this kind is homogeneous and isotropic as already explained above.

## 3.3 The Lorentz Process in the Poincaré half- plane

### 3.3.1 Description of the model.

Let us now consider the following mechanical model. A single particle moves in the Poincaré half-plane, where static circular obstacles are distributed according to a Poisson measure. At each instant $t$, the state of the particle is described by the couple $(q, v)$, where $q = (x, y) \in H_2$ is the position in the half-plane, and $v = (\cos\alpha, \sin\alpha)$ represents the direction of motion. Whenever the position $q$ of a particle lies outside the obstacles, the particle moves along the (unique) geodesic line tangent to $v$ at the point $q$. We assume that the particle has unit hyperbolic speed, namely the hyperbolic distance traveled in a time $t$ is equal to $t$. For any initial state $(q, v)$ at $t = 0$, the evolution of the particle position until the first collision is given by the geodesic flow $\Phi^{(q,v)}(t)$, for $t \geq 0$. The explicit expression of the geodesic flow is not essential now and will be given in the Appendix (formula (3.4.3)).

When a collision with an obstacle occurs, the particle is reflected on its surface.

In our model we assume that the particle performs a "specular reflection" in $H_2$. Now, in order to generalize the notion of specular reflection from $\mathbb{R}^2$ (where it is straightforward) to $H_2$, we recall these two basic facts.

The first one is that the measure of hyperbolic angles in the Poincaré half-plane corresponds to the measure performed by an Euclidean observer (this is not true in general for all the models of hyperbolic space, for instance the Klein disk model). The second one is that the angle between two geodesic lines coincides with the one formed by the corresponding Euclidean tangents at the point of incidence, as well as the angle between a geodesic line and a circle is the one detected by the respective tangent lines.

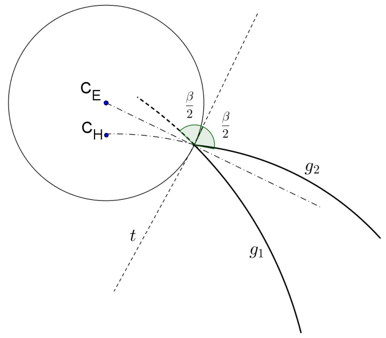


Figure 3.1: Specular deflection of angle $\beta$ due to an obstacle whose hyperbolic (Euclidean) center is $C_H$ ($C_E$). The angle of incidence and the angle of reflection are both equal to $\frac{\beta}{2}$.

Therefore, we refer to the specular reflection in $H_2$ in the following way: denoting respectively by $g_1$, $\gamma$ and $g_2$ the pre-collisional geodesic, the tangent to the obstacle and the post-collisional geodesic, we say that the particle is specularly reflected if the angle between $g_1$ and $\gamma$ is equal to the angle between $g_2$ and $\gamma$, as shown in figure (3.1).

Due to collisions, the sample paths of the moving particle are composed of arcs of circumferences pieced together (for an expression of the piecewise geodesic flow see the Appendix (formula (3.4.8)).

As a first step, assume now that a configuration of obstacle centers $\{c\} = \{c_1, c_2, ...c_j...\}$ is fixed. Let $r$ be the hyperbolic radius of the obstacles. For a given initial state $(q, v)$, the evolution of the particle position is given by the piecewice geodesic curve $\Phi^{(q,v)}_{\{c\}}(t)$, which clearly only depends on the obstacles of $\{c\}$ centered within a hyperbolic distance $t + r$ from $q$, since the hyperbolic velocity is assumed equal to 1. By deriving with respect to $t$ we obtain the Euclidean velocity of the particle (i.e. the velocity perceived by an Euclidean observer) which is denoted by $\dot{\Phi}^{(q,v)}_{\{c\}}(t)$. The direction of motion is given by the unit vector

$$V^{(q,v)}_{\{c\}}(t) = \frac{\dot{\Phi}^{(q,v)}_{\{c\}}(t)}{||\dot{\Phi}^{(q,v)}_{\{c\}}(t)||} \qquad t \geq 0 \tag{3.3.1}$$

where $||.||$ denotes the Euclidean norm. Hence we define the billiard flow (among the obstacles configuration $\{c\}$) as the following curve on the tangent bundle $H_2 \times S_1$:

$$\Psi^{(q,v)}_{\{c\}}(t) = \left(\Phi^{(q,v)}_{\{c\}}(t), V^{(q,v)}_{\{c\}}(t)\right) \qquad t \geq 0. \tag{3.3.2}$$

By assuming that the locations of the obstacles is random, the evolution of the particle defines a stochastic process $\{Q_r(t), V_r(t), t > 0\}$, on $H_2 \times S_1$ (the subscript "$r$" representing the radius of the obstacles) that we call hyperbolic Lorentz Process. We denote its joint density by $f_r(q, v, t)$ and suppose that an initial condition $f_r(q, v, 0) = f_{in}(q, v)$ is given, such that

$$\int_{H_2 \times S_1} f_{in}(q, v) dq dv = 1.$$

The function $f_{in}$ should be chosen in such a way that its support lies outside the system of obstacles, but here a difficulty arises since the obstacles location is random. We can skip this problem by choosing $f_{in}$ as any probability density on $H_2 \times S_1$ and assuming that if the particle initially lies inside an obstacle, it remains at rest forever. It is important to note that such a constraint disappears in the limit of small obstacles considered in this study.

For each $t > 0$, the joint density of the hyperbolic Lorentz Process is given by

$$f_r(q, v, t) = \mathbb{E}_{\{c\}} f_{in}\big(\Psi_{\{c\}}^{(q,v)}(-t)\big) \tag{3.3.3}$$

where the expectation is performed with respect to the Poisson measure.

Before stating the main result of the present work, it is necessary to determine the probability distribution of the free path length among Poissonian obstacles. The calculation requires some properties of hyperbolic geometry, and is shown in detail in the following section.

### 3.3.2 Free path among Poissonian obstacles

Let us consider a Poissonian distribution of spherical obstacles of hyperbolic radius $r$ in the Poincaré half-plane. Suppose that a particle, which is initially located at an arbitrary point $q \in H_2$, is shot towards an arbitrary direction $v$ and moves along the geodesic line tangent to $v$ at $q$. We are interested in the probability distribution of the first hitting time $T_{(q,v)}$ with the system of obstacles. Obviously, under the assumption of unitary hyperbolic speed, $T_{(q,v)}$ coincides with the free path length, namely with the hyperbolic distance traveled by the particle without having collisions.

The main idea is the following: the free path $T_{(q,v)}$ is greater than $t$ if and only if none of the obstacles has its center in the tube

$$\theta(q, v, t) = \left\{ p \in H_2 : \inf_{s \in [0,t]} d_h\big(p, \Phi^{(q,v)}(s)\big) < r \right\} \tag{3.3.4}$$

where $d_h(p, w)$ is the hyperbolic distance between two points $p$ and $w$ of the hyperbolic plane.

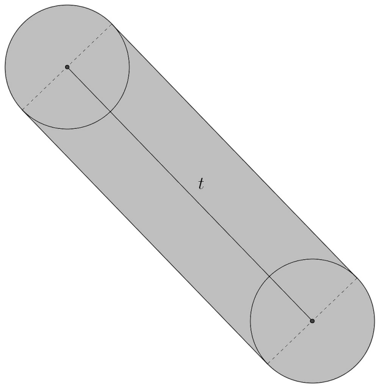


Figure 3.2: The Euclidean tube-like region $\theta(q, v, t)$ around the free particle trajectory.

In the Euclidean case, the tube is simply given by the union of a rectangle of sides $2r$ and $t$ and two half-circles of radius $r$ (see figure 3.2). Some difficulties arise in $H_2$, where, surprisingly, the two curves at hyperbolic distance $r$ on either side of a geodesic line are not geodesic lines.

We have then to determine the shape and the hyperbolic area of $\theta(q, v, t)$. To this aim we use the following representation:

$$\theta(q, v, t) = \bigcup_{0 \le s \le t} B_r(\Phi^{(q,v)}(s)). \tag{3.3.5}$$

In order to do that we make use of a suitable transformation in $H_2$. Indeed, one can show (see [14], Lemma 2.6) that among Möbius transformations (2.4.1) there exists a bijective isometry $\mathcal{M}_{(q,v)} : H_2 \to H_2$ such that[1] :

$$\mathcal{M}(\Phi^{(q,v)}(s)) = \Phi^{(\tilde{q},\tilde{v})}(s) \qquad \forall s \in \mathbb{R} \tag{3.3.6}$$

where $\tilde{q} = (0, 1)$ and $\tilde{v} = (0, 1)$, consequently $\Phi^{(\tilde{q},\tilde{v})}(s) = (0, e^s)$. In other words, $\mathcal{M}$ maps any geodesic line into a vertical geodesic line.

Through $\mathcal{M}$ the region $\theta(q, v, t)$ is mapped into the region:

$$\mathcal{M}(\theta(q, v, t)) = \left\{ w \in H_2 : \inf_{s \in [0,t]} d_h\big(\mathcal{M}^{-1}(w), \Phi^{(q,v)}(s)\big) < r \right\} =$$

$$= \left\{ w \in H_2 : \inf_{s \in [0,t]} d_h\big(w, \Phi^{(\tilde{q},\tilde{v})}(s)\big) < r \right\} = \theta(\tilde{q}, \tilde{v}, t) \tag{3.3.7}$$

where we used that $\mathcal{M}$ is invertible, it preserves distances and the property (3.3.6), so that the mapped region takes the following simple representation:

$$\theta(\tilde{q}, \tilde{v}, t) = \bigcup_{0 \le s \le t} B_r\big((0, e^s)\big). \tag{3.3.8}$$

[1] We omit the calculations for sake of brevity. $\mathcal{M}_{(q,v)}$ depends on $(q, v)$ as parameters; for simplicity we will use the notation $\mathcal{M}$ in the following.

Moreover since $\mathcal{M}$ is an isometry, it preserves areas, whence we can finally compute the desired area as:

$$|\theta(q,v,t)| = |\theta(\tilde{q},\tilde{v},t)| \tag{3.3.9}$$

Now, (3.3.8) is the region inside the envelope of the following family of curves

$$C_t = \{\partial B_r\big((0,s)\big),\quad 1 \leq s \leq e^t\} \tag{3.3.10}$$

where $\partial B_r\big((0,s)\big)$ has cartesian equation

$$h(x,y,s) = x^2 + (y - s\cosh r)^2 - s^2\sinh^2 r = 0.$$

We obtain the envelope of $C_t$ by means of the following system

$$\begin{cases} h(x,y,s) = 0 \\ \frac{\partial}{\partial s} h(x,y,s) = 0 \end{cases} \tag{3.3.11}$$

which gives the union of the following Euclidean lines

$$y = \frac{x}{\sinh r} \qquad\qquad y = -\frac{x}{\sinh r} \tag{3.3.12}$$

Thus, the tube $\theta(\tilde{q},\tilde{v},t)$ is the section of a cone with vertex in $(0,0)$ and central axis the line $x = 0$, as shown in figure 3.3.

It is now important to observe that (3.3.12) is tangent to $\partial B_r(0,1)$ at the points $A = (\tanh r; \frac{1}{\cosh r})$ and $B = (-\tanh r, \frac{1}{\cosh r})$, and also tangent to $\partial B_r(0,e^t)$ at the points $C = (-e^t\tanh r; \frac{e^t}{\cosh r})$ and $D = (e^t\tanh r, \frac{e^t}{\cosh r})$.

Moreover $A$ and $B$ lie on the geodesic line $x^2+y^2 = 1$, while $C$ and $D$ lie on $x^2+y^2 = e^{2t}$. This makes it clear that $\theta(\tilde{q},\tilde{v},t)$ is composed of three parts: the half-circle below the geodesic segment $AB$, the intermediate region $\theta'(\tilde{q},\tilde{v},t)$ with vertices $A, B, C, D$ and the half-circle above the geodesic segment $CD$. The hyperbolic area of $\theta'(\tilde{q},\tilde{v},t)$ is defined as

$$|\theta'(\tilde{q},\tilde{v},t)| = \int_{\theta'(\tilde{q},\tilde{v},t)} \frac{dxdy}{y^2}$$

By means of the substitutions $x = \rho\cos\gamma$ and $y = \rho\sin\gamma$ we have that

$$|\theta'(\tilde{q},\tilde{v},t)| = \int_1^{e^t}\int_{\alpha}^{\pi-\alpha} \frac{\rho\, d\rho d\gamma}{(\rho\sin\gamma)^2} = \frac{2t}{\tan\alpha}$$

where $\tan\alpha = \frac{1}{\sinh r}$ is related to the slope of (3.3.12). Finally, the area of $\theta(\tilde{q},\tilde{v},t)$ is given by

$$|\theta(\tilde{q},\tilde{v},t)| = 4\pi\sinh^2\frac{r}{2} + 2t\sinh r \tag{3.3.13}$$

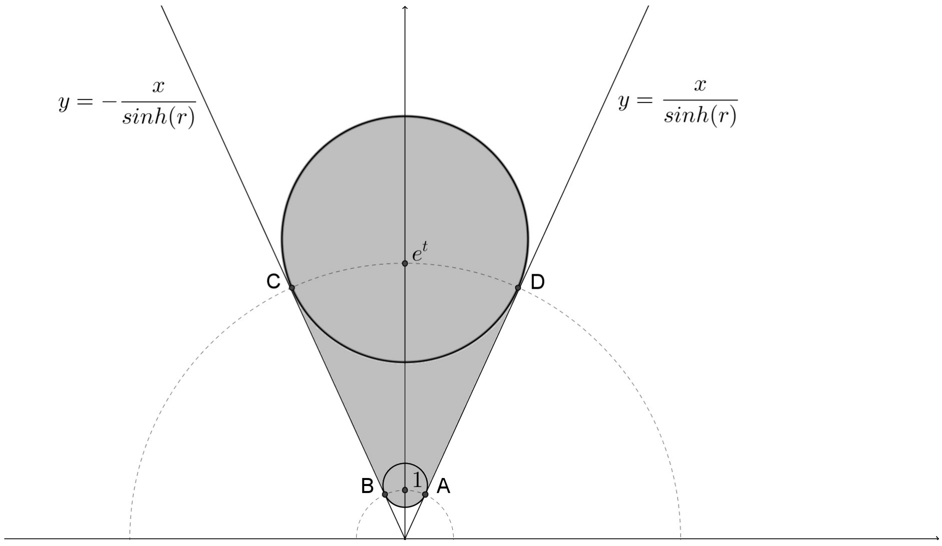


Figure 3.3: The hyperbolic tube-like region $\theta(\tilde{q}, \tilde{v}, t)$ around the free particle trajectory.

We can now come back to the probability distribution of the first hitting time $T_{(q,v)}$. As before, we denote by $N(\mathcal{S})$ the number of points inside $\mathcal{S}$. By excluding the possibility that $q$ is located within some obstacles, we thus have

$$\Pr\left\{T_{(q,v)} > t \,|N(B_r(q)) = 0\right\} = \frac{\Pr\left\{T_{(q,v)} > t\,,\, N(B_r(q) = 0\right\}}{\Pr\left\{N(B_r(q) = 0\right\}}$$

$$= \frac{e^{-\lambda|\theta(q,v,t)|}}{e^{-\lambda|B_r(q)|}} = e^{-2\lambda t \sinh r}$$

where, in the last equality, we used (3.3.9) and (**??**).
We conclude that $T_{(q,v)}$ has an exponential probability distribution with parameter $2\lambda \sinh r$. Thus the mean free path, which is a fundamental quantity in what follows, is given by

$$\sigma^{-1} = (2\lambda \sinh r)^{-1}. \tag{3.3.14}$$

Performing the same calculation in the Euclidean case leads to say that the free path has an exponential distribution of parameter $2\lambda r$ and the mean free path is thus $(2\lambda r)^{-1}$.

### 3.3.3 The main theorem

The most important result of the present work is the next theorem, where we find a suitable scaling limit corresponding to small obstacles. Among all possible scalings, the one consisting in

$$r \to 0 \qquad \lambda \to \infty \qquad \text{in such a way that} \quad 2\lambda \sinh r = \sigma > 0,$$

that we call hyperbolic Boltzmann-Grad limit in analogy with Gallavotti's work, ensures a non-trivial approximation for $f_r(q, v, t)$.

Moreover, as a final result, we will also show (see section 3.4) that the limit function $f(q, v, t)$ is the probability density of a Markovian process, namely a random flight $\{(Q(t), V(t)), t > 0\}$ on the Poincaré half plane.

**Theorem 2.** *Let $\{(Q_r(t), V_r(t)), t > 0\}$ be the Lorentz process in the Poincaré half-plane, defined in such a way that the obstacles are disks of hyperbolic radius $r$, whose centers are distributed as a hyperbolic homogeneous Poisson field with intensity $\lambda = \frac{\sigma}{2\sinh r}$. Let $f_{in} \in L_\infty(H_2 \times S_1)$ be the initial probability density. Then, in the limit $r \to 0$, the joint density $f_r$ of the Lorentz process converges in $L_1$ sense to some probability density $f$ for each $t > 0$. Moreover $f$ solves the following equation*

$$\frac{\partial}{\partial t} f(q, v, t) + \mathcal{D} f(q, v, t) = -\sigma f + \sigma \int_0^{2\pi} f(q, R_\beta v, t) \frac{1}{4} \sin\frac{\beta}{2} d\beta \qquad (3.3.15)$$

$$f(q, v, 0) = f_{in}(q, v),$$

*where $\mathcal{D}$ is the operator of covariant differentiation along the geodesic lines and $R_\beta$ is the rotation of an angle $\beta$.*

*Proof.* It is possible to write explicitly $f_r(q, v, t)$. Of course, $f_r(q, v, t) = f_{in}(q, v)$ if $q$ lies inside any obstacle. The following calculations are made on condition that no obstacle center lies inside the ball of hyperbolic radius $r$ around $q$.

Let us suppose that the particle state at time $t$ is given by $(q, v)$ and consider the backward trajectory. We denote by $N_{(q,v)}(t)$ the number of collisions which occurred up to time $t$. From section 3.2, it is clear that

$$\Pr\left\{N_{(q,v)}(t) = 0\right\} = \Pr\left\{T_{(q,v)} > t\right\} = e^{-2\lambda t \sinh r}. \qquad (3.3.16)$$

Instead, the probability that the particle collides exactly $n$ obstacles whose centers are located in the infinitesimal hyperbolic areas $dc_1, ....dc_n$ around $c_1, ....c_n$ is

$$\Pr\left\{N_{(q,v)}(t) = n, C_1 \in dc_1, \dots, C_n \in dc_n\right\} = \lambda^n e^{-\lambda|\theta_{\{c\}}(q,v,t)|} dc_1 \cdots dc_n \qquad (3.3.17)$$

where $\theta_{\{c\}}(q, v, t)$ is the tube of hyperbolic width $2r$ around the particle trajectory, namely the tube-like region swept by an ideal obstacle when its center is moved along the path. While in the Euclidean case this region is simply given by a non disjoint union of rectangles, it here has a more complex shape and its hyperbolic area can be estimated as

$$|\theta_{\{c\}}(q, v, t)| = 2t \sinh r + o(\sinh r). \qquad (3.3.18)$$

The particle density (3.3.3) can be written as

$$\begin{aligned} f_r(q,v,t) &= f_{in}(\Psi^{(q,v)}(-t))e^{-2\lambda t \sinh r} \\ &\quad + \sum_{n=1}^{\infty} \int_{A^n_{q,v}} f_{in}(\Psi^{(q,v)}_{(c_1 \cdots c_n)}(-t))\lambda^n e^{-\lambda|\theta_{\{c\}}(q,v,t)|} dc_1 \cdots dc_n \end{aligned} \tag{3.3.19}$$

where $\Psi$ denotes the billiard flow defined in (3.3.2) and $A^n_{q,v}$ is the subset of $B^n_{t+r}(q)$, containing all the obstacles configurations such that the backward trajectory with initial state $(q,v)$ collides with the $n$ obstacles centered at $c_1 \cdots c_n$.

Following Gallavotti's proof, we observe that among all the possible configurations of obstacles, there are some such that the trajectory hits each obstacle at most once, and there are others that lead to recollisions. Thus we can split $f_r$ into two components, that we respectively call the "Markovian" and the "recollision" terms:

$$f_r(q,v,t) = f_r^M(q,v,t) + f_r^{REC}(q,v,t). \tag{3.3.20}$$

We now restrict our attention to the Markovian term $f_r^M(q,v,t)$. By considering the backward evolution, let $\tau_1 \cdots \tau_n$ be the collision times, such that

$$0 < \tau_1 < \tau_2 \cdots < \tau_n < t \tag{3.3.21}$$

and $\beta_1 \cdots \beta_n$ be the corresponding deflection angles. Then, for a given particle trajectory, there is a one-to one correspondence between the $2n$ variables $c_1 \cdots c_n$ and the $2n$ variables $\tau_1 \cdots \tau_n, \beta_1 \cdots \beta_n$.

We now obtain the transformation

$$dc_1 ... dc_n = \frac{1}{2^n} \sinh^n(r) \sin\frac{\beta_1}{2} ... \sin\frac{\beta_n}{2} d\tau_1 ... d\tau_n d\beta_1 ... d\beta_n \tag{3.3.22}$$

which is a crucial point to prove the theorem. For simplicity, we treat the case $n = 1$, the general case follows immediately by carrying out tedious calculations. We explain how to determine the center $C = (x_C, y_C)$ of the obstacle in terms of the flight time $\tau$ and the deflection angle $\beta$.

Let the particle be at position $q$ and unit velocity $v$ at time $t$; we are interested in its backward evolution . After a time $\tau$ the particle collides with the first obstacle, namely a circle centered at $C = (x_C, y_C)$, and it is reflected in a trajectory forming an angle $\beta$ with the incoming trajectory as shown in figure 3.1.
In order to perform the calculation we first employ the Möbius transformation (3.3.6). In this way the particle backward trajectory is mapped into $\Phi^{(\tilde{q},\tilde{v})}(s) = (0, e^s)$ for $s \in [0, \tau)$; besides, the obstacle centered at $C = (x_C, y_C)$ is mapped into an obstacle centered at $\tilde{C} = (\tilde{x_C}, \tilde{y_C})$ of the same hyperbolic radius[2].

[2]Since $\mathcal{M}$ preserve distances, it sends circumferences into circumferences.

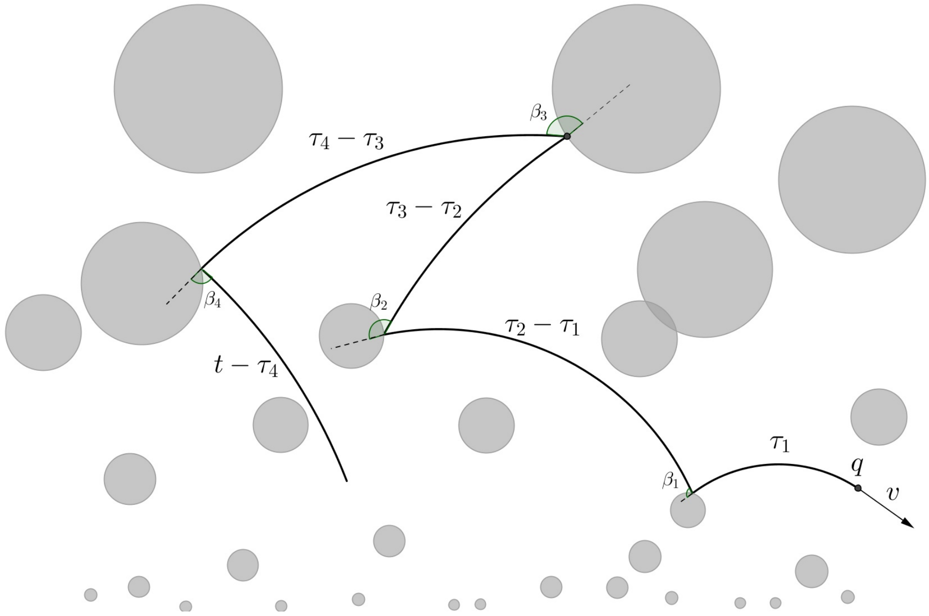


Figure 3.4: A typical backward trajectory with four collisions, starting from the state $(q, v)$ at time $t$.

The traveled time $\tau$ is preserved under an isometry, thus the point of impact is mapped into $(0, e^\tau)$; the deflection angle $\beta$ is also preserved since $\mathcal{M}$ is conformal (as it is an isometry). These facts will be of great importance in the following calculation.
Finally we are interested in the area element $dc$ that can be computed as:

$$dc = d\tilde{c} = \frac{d\tilde{x_C}\, d\tilde{y_C}}{\tilde{y_C}^2} \tag{3.3.23}$$

In this setting, as can be clearly seen in figure 3.1, one can ideally "reach" the obstacle center $\tilde{C} = (\tilde{x_C}, \tilde{y_C})$ from the collision point $(0, e^\tau)$; it is sufficient to rotate the unit velocity of an angle $\gamma = \frac{\pi}{2} - \frac{\beta}{2}$ and to travel along a path of hyperbolic length $r$. Thus, by using (3.4.3) we have

$$\tilde{x_C} = -\frac{e^\tau \sinh r \cos\frac{\beta}{2}}{\cosh r - \sin\frac{\beta}{2}\sinh r} \tag{3.3.24}$$

$$\tilde{y_C} = \frac{e^\tau}{\cosh r - \sin\frac{\beta}{2}\sinh r}. \tag{3.3.25}$$

The Jacobian matrix of the trasformation $(\tilde{x}_C, \tilde{y}_C) \to (\tau, \beta)$ is given by

$$J = \begin{pmatrix} \frac{-e^\tau \sinh r \cos\frac{\beta}{2}}{\cosh r - \sin\frac{\beta}{2}\sinh r} & \frac{1}{2}\frac{e^\tau \sinh r \cosh r \sin\frac{\beta}{2} - e^\tau \sinh^2 r}{(\cosh r - \sin\frac{\beta}{2}\sinh r)^2} \\ \frac{e^\tau}{\cosh r - \sin\frac{\beta}{2}\sinh r} & \frac{1}{2}\frac{e^\tau \cos\frac{\beta}{2}\sinh r}{(\cosh r - \sin\frac{\beta}{2}\sinh r)^2} \end{pmatrix}$$

and its determinant reads

$$\det J = \frac{1}{2}\frac{e^{2\tau}\sinh r \sin\frac{\beta}{2}}{(\cosh r - \sin\frac{\beta}{2}\sinh r)^2} \tag{3.3.26}$$

Therefore the infinitesimal surface element is given by

$$d\tilde{c} = \frac{d\tilde{x_C}\, d\tilde{y_C}}{\tilde{y_C}^2} = \frac{1}{2}\sinh r\ \sin\frac{\beta}{2}\, d\beta d\tau. \tag{3.3.27}$$

as desired. Thus the Markovian term can be then written as

$$\begin{aligned} f_r^M(q,v,t) = f_{in}(\Psi^{(q,v)}(-t))e^{-2\lambda t \sinh r} + \sum_{n=1}^{\infty}\int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 .... \int_{\tau_{n-1}}^t d\tau_n \\ \int_{[0,2\pi]^n} d\beta_1 ....d\beta_n\, f_{in}(\Psi^{(q,v)}_{\tau_1...\tau_n,\beta_1...\beta_n}(-t)) \\ e^{-\lambda|\theta_{\{c\}}(q,v,t)|}\frac{\lambda^n}{2^n}\sinh^n r \sin\frac{\beta_1}{2} ... \sin\frac{\beta_n}{2} \end{aligned} \tag{3.3.28}$$

.

Observe that the general term in the summation is bounded by

$$||f_{in}||_{L_\infty}\frac{(\sigma t)^n}{n!}$$

which is the $n^{th}$ term of an exponential converging series.

Thus, in force of the dominated convergence theorem for the series, passing to the limit as $r \to 0$, together with the assumption $\lambda = \frac{\sigma}{2\sinh r}$, we have that $f_r^M(q,v,t)$ converges pointwise to

$$\begin{aligned} f(q,v,t) =& f_{in}(\Psi^{(q,v)}(-t))e^{-\sigma t} + e^{-\sigma t}\sum_{n=1}^{\infty}\sigma^n \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 .... \int_{\tau_{n-1}}^t d\tau_n \\ & \int_{[0,2\pi]^n} d\beta_1 ....d\beta_n\, \frac{1}{4^n}\sin\frac{\beta_1}{2} ... \sin\frac{\beta_n}{2} f_{in}(\Psi^{(q,v)}_{\tau_1...\tau_n,\beta_1...\beta_n}(-t)) \end{aligned} \tag{3.3.29}$$

and the limit clearly holds also in $L_1$ sense.

By easy calculations we can check that (3.3.29) is a probability density, since

$$||f||_{L_1} = \int_{H_2\times S_1} f(q,v,t)dqdv = 1 \tag{3.3.30}$$

and furthermore it is bounded in the following way:

$$||f||_{L_\infty} \le ||f_{in}||_{L_\infty} \tag{3.3.31}$$

From (3.3.20) we have

$$\int_{H_2\times S_1} f_r(q,v,t)dqdv = \int_{H_2\times S_1} f_r^M(q,v,t)dqdv + \int_{H_2\times S_1} f_r^{REC}(q,v,t)dqdv \quad (3.3.32)$$

Passing to the Boltzmann-Grad limit, it is clear that the left hand side of (3.3.32) is equal to 1, because the billiard flow obviously leaves the total probability mass invariant. Moreover, conditions (3.3.30) and (3.3.31) and the dominated convergence theorem lead to

$$1 = 1 + \lim_{r\to 0}\int_{H_2\times S_1} f_r^{REC}(q,v,t)dqdv \quad (3.3.33)$$

The contribution of the recollision term thus vanishes in $L_1$-norm, namely the measure of all the pathological paths goes to zero. In the end, by writing (3.3.20) as

$$f_r(q,v,t) - f(q,v,t) = f_r^M(q,v,t) - f(q,v,t) + f_r^{REC}(q,v,t) \quad (3.3.34)$$

immediately follows that

$$||f_r(q,v,t) - f(q,v,t)||_{L_1} \leq ||f_r^M(q,v,t) - f(q,v,t)||_{L_1} + ||f_r^{REC}(q,v,t)||_{L_1} \quad (3.3.35)$$

Under the Boltzmann- Grad limit, the right side of (3.3.35) vanishes and the proof of convergence is complete.

To prove that (3.3.29) is a solution to (3.3.15), we need to define two kinds of operators, which are bounded in the norm $||.||_{L_\infty}$. The first one is the semigroup of geodesic transport with damping, given by

$$T_t f(q,v) = e^{-\sigma t} f(\Phi^{(q,v)}(-t), V^{(q,v)}(-t))$$

which is generated by

$$A = -\sigma - \mathcal{D}$$

where $\mathcal{D}$ denotes the operation of differentiation along the curve $(\Phi^{(q,v)}(t), V^{(q,v)}(t))$ lying in $H_2 \times S_1$. The second one is the collision operator:

$$Lf(q,v) = \sigma \int_0^{2\pi} f(q, R_\beta v)\frac{1}{4}\sin\frac{\beta}{2}d\beta.$$

Thus (3.3.29) can be written as

$$f(q,v,t) = T_t f_{in}(q,v) + \sum_{n=1}^{\infty}\int_{0\leq s_1\leq\ldots\leq s_n\leq t} T_{t-s_n} L T_{s_n - s_{n-1}} \ldots L T_{s_1} f_{in}(q,v) ds_1 \ldots ds_n$$

which is the Duhamel expansion giving (see [39], page 52) the solution to

$$\frac{\partial}{\partial t} f = (A+L)f$$

and this concludes the proof. □

### 3.3.4 The limit Markovian random flight.

We now show that a Markovian transport process $\{(Q(t), V(t)), t > 0\}$ exists whose finite one-dimensional distribution is given by (3.3.29). The rigorous construction of a process of this kind can be carried out by following the same steps as Pinsky [63], who defined a random flight on the tangent bundle of a generic Riemannian manifold. We assume that $Q(0) = \tilde{q}$ and $V(0) = \tilde{v}$ almost surely, the case of distributed initial data is an immediate consequence. We denote such a process by $\left(Q^{(\tilde{q},\tilde{v})}(t), V^{(\tilde{q},\tilde{v})}(t)\right)$ and we outline its construction. Let us consider a sequence of independent waiting times $e_j$, $j \geq 1$, having distribution

$$\Pr\{e_j > \eta\} = e^{-\sigma\eta} \qquad \eta > 0.$$

A particle moves along the geodesic lines in $H_2$ and changes direction at Poisson times

$$\tau_n = e_1 + e_2 + \cdots e_n \qquad n \geq 1. \tag{3.3.36}$$

For $0 \leq t \leq \tau_1$ we have

$$Q^{(\tilde{q},\tilde{v})}(t) = \Phi^{(\tilde{q},\tilde{v})}(t) \qquad\qquad V^{(\tilde{q},\tilde{v})}(t) = \frac{\dot{\Phi}^{(\tilde{q},\tilde{v})}(t)}{||\dot{\Phi}^{(\tilde{q},\tilde{v})}(t)||}, \tag{3.3.37}$$

where $\Phi$ is the geodesic flow defined in (3.4.3). The random point where the first deflection occurs and the corresponding post-collisional velocity are respectively given by

$$Q_1 = \Phi^{(\tilde{q},\tilde{v})}(\tau_1) \qquad\qquad V_1 = R_{(\beta_1)}\left[V^{(\tilde{q},\tilde{v})}(\tau_1^-)\right] \tag{3.3.38}$$

where $R_{(\beta)}$ denotes the rotation of an angle $\beta$. Proceeding recursively, the process is such that, for $\tau_n \leq t \leq \tau_{n+1}$

$$Q^{(\tilde{q},\tilde{v})}(t) = \Phi^{(Q_n,V_n)}(t-\tau_n) \qquad\qquad V^{(\tilde{q},\tilde{v})}(t) = \frac{\dot{\Phi}^{(Q_n,V_n)}(t-\tau_n)}{|\dot{\Phi}^{(Q_n,V_n)}(t-\tau_n)|} \tag{3.3.39}$$

where the random point of the $j^{th}$ deflection and the $j^{th}$ post-collisional velocity are denoted by

$$Q_j = \Phi^{(Q_{j-1},V_{j-1})}(e_j) \qquad\qquad V_j = R_{(\beta_j)}\left[V^{(Q_{j-1},V_{j-1})}(e_j^-)\right] \tag{3.3.40}$$

A crucial point of the construction is that the deflection angles $\beta_j$ are independent of the Poissonian times and among themselves; they have common distribution

$$\Pr\{\theta_j \in d\beta\} = \frac{1}{4}\sin\frac{\beta}{2}d\beta \qquad \beta \in [0, 2\pi]$$

and this is consistent with the cross section due to the collision with a hard sphere.

In the case where the process has initial density $f_{in}(q, v)$, the single-time density of $\{(Q(t), V(t)), t > 0\}$ is just given by (3.3.29) and it can be written as

$$f(q, v, t) = \tilde{T}_t f_{in}(q, v) = \mathbb{E}\{f_{in}(Q^{(q,v)}(t), V^{(q,v)}(t))\}$$

where $\{\tilde{T}_t, t > 0\}$ is a strongly continuous contraction semigroup on the space of differentiable and bounded functions on $H_2 \times S_1$, endowed with the norm $||.||_{L\infty}$. Then, following the same steps of Pinsky [63], we can write the generator of $\tilde{T}_t$: the limit density (3.3.29) satisfies the following linear Boltzmann-type differential equation (obviously coinciding with 3.3.15)

$$\frac{\partial}{\partial t} f(q, v, t) + \mathcal{D} f(q, v, t) = \sigma \int_0^{2\pi} \left(f(q, R_\beta v, t) - f(q, v, t)\right) \frac{1}{4} \sin \frac{\beta}{2} d\beta$$

where $\mathcal{D}$ denotes the operator of covariant differentiation along a geodesic line and $R_\beta$ is the rotation of an angle $\beta$ .

### 3.3.5 Further remarks

In this thesis we studied the Lorentz process and the related Boltzmann-Grad limit on a classical model of hyperbolic geometry, namely the Poincaré half-plane. It is not straightforward to show that our results hold in any hyperbolic space. Some problems could arise, for example, when calculating the mean free path, as this is based on the knowledge of the volume of the tube-like regions.

More precisely, it would be interesting to study the Lorentz process on another well-known hyperbolic manifold, namely the Poincaré disk $\mathbb{D}_2$, which we recall to be defined as the set $\mathbb{D}_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ endowed with the metric

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}. \tag{3.3.41}$$

Equivalently, $\mathbb{D}_2$ can be defined as the complex domain $\mathbb{D}_2 = \{z \in \mathbb{C} : |z| < 1\}$ with infinitesimal arc-length given by $\frac{2|dz|}{1-|z|^2}$. We believe that the isomorphism between $H_2$ and $\mathbb{D}_2$ could be employed to study the Lorentz process in this space. The isomorphism is given by the Cayley transform $\mathcal{K} : H_2 \to \mathbb{D}_2$ that maps a point $z \in H_2$ into a point $w \in \mathbb{D}_2$:

$$w = \frac{iz + 1}{z + i}. \tag{3.3.42}$$

If a geodesic line in $H_2$ is represented by an euclidean half circle with center $(x_0, 0)$ and radius $r$, its image through $\mathcal{K}$ is given by an arc of circumference which is orthogonal to the border of $\mathbb{D}_2$, having center at $\left(\frac{2x_0}{x_0^2-r^2+1}, \frac{x_0^2-r^2-1}{x_0^2-r^2+1}\right)$ and radius $R$ such that $R^2 = (\frac{2r}{x_0^2-r^2+1})^2$.

Now, although $\mathcal{K}$ represents a contraction of the half-plane into the unitary disk, it is a conformal transform, namely it leaves angles between geodesic lines invariant and we saw how the measure of scattering angles plays a fundamental role in the study of the process. The previous observation, together with the fact that $\mathcal{K}^{-1}$ sends geodesic lines in $\mathbb{D}_2$ into geodesic lines in $H_2$ suggests that a suitable use of the Cayley transform could be the main tool in the study of the Lorentz process in $\mathbb{D}_2$.

We remark that the assumption that hyperbolic angles coincide with the angles measured by an Euclidean observer doesn't hold for example in the Klein disc model for hyperbolic space (for random motions with branching on the Klein disk see, for example, [33]).

Finally, we would like to state that we did not define the most general Lorentz process on the hyperbolic half-plane. One could investigate, for example, the case of randomly moving obstacles (as Desvillettes and Ricci did in [21] in an Euclidean context): it would be reasonable to assume that each obstacle moves with a fixed hyperbolic velocity, following a Gaussian distribution.

Another line of research could be the analysis of the Lorentz model when other boundary conditions are assumed, e.g. the particle could be re-emitted with a stochastic law instead of being specularly reflected by the obstacles.

## 3.4 Appendix. Piecewise geodesic motion on the hyperbolic half-plane.

Before writing an explicit expression for the geodesic flow in the Poincaré half-plane, we recall the corresponding one in the Euclidean context. If a particle starts at $q \in \mathbb{R}^2$ with velocity $v$ and is not subject to collisions, the position at time $t$ is well known to be equal to

$$\Phi^{(q,v)}(t) = q + vt \tag{3.4.1}$$

On the other hand, let $\tau_1 \dots \tau_n$ be the hitting times, with

$$0 < \tau_1 < \tau_2 < \cdots < \tau_n < t \tag{3.4.2}$$

and let $\beta_1 \dots \beta_n$ be the corresponding deflection angles. The position of the particle at time $t$, that we call $\Phi_n^{(q,v)}(t)$ for simplicity of notation, can be written as

$$\Phi_n^{(q,v)}(t) = q + v\tau_1 + v_1(\tau_2 - \tau_1) + \cdots + v_{n-1}(\tau_n - \tau_{n-1}) + v_n(t - \tau_n)$$

with

$$v_j = R_{\beta_j} v_{j-1} \qquad\qquad v_0 = v$$

$R_{\beta_j}$ representing the matrix of rotation of an angle $\beta_j$.

We now consider a particle starting at $q = (x_0, y_0) \in H_2$ with velocity $v = (\cos\alpha, \sin\alpha)$. Suppose that the particle moves along the geodesic line with hyperbolic velocity of intensity 1. Then, the position of the particle at time $t$ is given by

$$\Phi^{(q,v)}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 + y_0 \frac{\sinh t \cos\alpha}{\cosh t - \sin\alpha \sinh t} \\ \frac{y_0}{\cosh t - \sin\alpha\sinh t} \end{pmatrix} \tag{3.4.3}$$

We now show how to obtain formula (3.4.3). Observe that $(x(t), y(t))$ is obviously given by the intersection of 2 curves, namely the hyperbolic circle of radius $t$ centered at $q = (x_0, y_0)$, having equation

$$(x - x_0)^2 + (y - y_0 \cosh t)^2 = y_0^2 \sinh^2 t \tag{3.4.4}$$

and the geodesic line tangent to $v$ at the point $q$, having equation

$$(x - x_0 - y_0 \tan\alpha)^2 + y^2 = \frac{y_0^2}{\cos\alpha^2}. \tag{3.4.5}$$

This corresponds to the following system

$$\begin{cases} (X - \tan\alpha)^2 + Y^2 = \frac{1}{\cos^2\alpha} \\ X^2 + (Y - \cosh t)^2 = \sinh^2 t \end{cases}$$

with $X = \frac{x - x_0}{y_0}$ and $Y = \frac{y}{y_0}$, whose solution is

$$\begin{cases} X = \frac{\sinh t \cos\alpha}{\cosh t - \sin\alpha \sinh t} \\ Y = \frac{1}{\cosh t - \sin\alpha\sinh t} \end{cases}$$

and (3.4.3) is proved.

By deriving ( 3.4.3) with respect to $t$ we obtain the Euclidean velocity (i.e. the velocity perceived by an Euclidean observer):

$$\dot\Phi^{(q,v)}(t) = \begin{pmatrix} \dot x(t) \\ \dot y(t) \end{pmatrix} = \frac{y_0}{(\cosh t - \sin\alpha\sinh t)^2} \begin{pmatrix} \cos\alpha \\ -\sinh t + \sin\alpha\cosh t \end{pmatrix} \tag{3.4.6}$$

which is a parallel vector field along the curve $\Phi^{(q,v)}(t)$, whose norm is given by $||\dot{\Phi}^{(q,v)}(t)|| = y(t)$. The unit velocity vector is given by

$$V^{(q,v)}(t) = \frac{\dot{\Phi}^{(q,v)}(t)}{||\dot{\Phi}^{(q,v)}(t)||} = \frac{1}{\cosh t - \sin\alpha \sinh t} \begin{pmatrix} \cos\alpha \\ -\sinh t + \sin\alpha \cosh t \end{pmatrix}. \tag{3.4.7}$$

We now consider the case where the path is a piecewise geodesic. Let $\tau_1 ... \tau_n$ be the hitting times and $\beta_1 ... \beta_n$ be the corresponding deflection angles. The particle starts at $q = (x_0, y_0)$ with velocity $v = (\cos\alpha, \sin\alpha)$, then it travels along the geodesic line until a time $\tau_1$, when the position is $\Phi^{(q,v)}(\tau_1)$ and the unit velocity is given by the vector $V^{(q,v)}(\tau_1^-)$, which changes into $v_1 = (\cos\alpha_1, \sin\alpha_1) = R_{\beta_1} V^{(q,v)}(\tau_1^-)$. During the time interval $[\tau_{j-1}, \tau_j]$ the velocity evolves from $v_{j-1}$ to $V(\tau_j^-)$ and, at time $\tau_j$ it is changed to $v_j = (\cos\alpha_j, \sin\alpha_j) = R_{\beta_j} V(\tau_j^-)$. By iterating (3.4.3) we immediately obtain

$$\Phi_n^{(q,v)}(t) = \begin{pmatrix} x(\tau_n) + y(\tau_n) \frac{\sinh(t-\tau_n)\cos\alpha_n}{\cosh(t-\tau_n) - \sin\alpha_n \sinh(t-\tau_n)} \\ y(\tau_n) \cdot \frac{1}{\cosh(t-\tau_n) - \sin\alpha_n \sinh(t-\tau_n)} \end{pmatrix} \tag{3.4.8}$$

for $\tau_n < t < \tau_{n+1}$, where $x(\tau_n)$ and $y(\tau_n)$ can be computed in a recursive way by using (3.4.3).

# Chapter 4

# Lévy processes and subordinators.

This chapter contains some basic facts on Lévy processes, with particular attention to the subclass of subordinators. For a complete reference see [64] and [2].

## 4.1 Infinite divisibility

**Definition 1.** A random variable $X$ is said to be *infinitely divisible* if for each $n \in \mathbb{N}$ there exists a sequence of i.i.d random variables $Y_1$, $Y_2$, ... $Y_n$, such that

$$X \stackrel{d}{=} Y_1 + Y_2 + ... + Y_n. \tag{4.1.1}$$

Denoting respectively by $\mu$ and $\mu^{\frac{1}{n}}$ the probability distribution of $X$ and each $Y_j$, formula 4.1.1 reduces to the following convolution

$$\mu = \mu^{\frac{1}{n}} * \mu^{\frac{1}{n}} * ..... * \mu^{\frac{1}{n}} \tag{4.1.2}$$

Moreover let $\phi(u) = \mathbb{E}e^{iuX}$ be the characteristic function of $X$, then the characteristic function of each $Y_j$ obviously reduces to $(\phi(u))^{\frac{1}{n}}$, and this furnishes a manageable criterion to recognize infinitely divisible random variables, as the following examples show.

**Example 1 (Compound Poisson)** Let $N$ be a Poisson random variable with mean $\lambda$ and let $Z_j$, $j \geq 1$ be a sequence of i.i.d random variables, each of them having probability distribution $\psi_Z$. Then

$$X = \sum_{j=1}^{N} Z_j \tag{4.1.3}$$

is said to be a *Compound Poisson* random variable and its distribution is denoted by $CP(\lambda, \psi_Z)$. The characteristic function of 4.1.3 is

$$\begin{aligned}\phi(u) &= \mathbb{E}e^{iuX} = \mathbb{E}_N\mathbb{E}e^{iu\sum_{j=1}^{N} Z_j} = \mathbb{E}_N(\mathbb{E}e^{iuZ_j})^N \\ &= \sum_{n=0}^{\infty}(\mathbb{E}e^{iuZ_j})^n e^{-\lambda}\frac{\lambda^n}{n!} = e^{-\lambda+\lambda\mathbb{E}e^{iuZ_j}} \\ &= e^{-\lambda\int_{-\infty}^{\infty}(e^{iux}-1)\psi_Z(dx)}\end{aligned} \tag{4.1.4}$$

Now, the $n$-th root of (4.1.4) is the characteristic function of $CP(\frac{\lambda}{n}, \psi_Z)$. So, if $X$ has distribution $CP(\lambda, \psi_Z)$, it is infinitely divisible and, for each $n \in \mathbb{N}$, definition (4.1.1) holds by taking each $Y_j$ with distribution $CP(\frac{\lambda}{n}, \psi_Z)$ .

When $Z_j \stackrel{d}{=} 1\ \forall j$, namely $\psi_Z(dz) = \delta_1(dz)$, the Compound Poisson distribution reduces to a Poisson distribution with rate $\lambda$, which is infinitely divisible and (4.1.1) holds by taking each $Y_j$ with poissonian distribution with rate $\frac{\lambda}{n}$

**Example 2 (Gaussian distribution)** If $X$ is a gaussian random variable with mean $\mu$ and variance $\sigma^2$, then its characteristic function is $\phi(u) = e^{iu\mu+\frac{1}{2}\sigma^2u^2}$. Its $n$-th root is the characteristic function of a gaussian random variable with mean $\frac{\mu}{n}$ and variance $\frac{\sigma^2}{n}$. Thus $X$ is infinitely divisible.

A fundamental result about infinitely divisible random variables is the Lévy - Khintchine formula, which preliminary requires the following definition:

**Definition 2.** A measure $\nu$ on $\mathbb{R}$ is a Lévy measure if

$$\nu(0) = 0 \qquad \int_{\mathbb{R}-\{0\}}(y^2 \wedge 1)\nu(dy) < \infty. \tag{4.1.5}$$

We now state the Lévy -Khintchine theorem, and we refer to [2] for the proof.

**Theorem 3.** *Let $X$ be an infinitely divisible random variable. Then there exists two constants $b$ and $\sigma$ and a Lévy measure $\nu$ such that the characteristic function of $X$ reads*

$$\phi(u) = e^\wedge\left(ibu - \frac{1}{2}\sigma^2u^2 + \int_{\mathbb{R}-\{0\}}(e^{iuy} - 1 - iuy\,\chi_{[-1,1](y)})\nu(dy)\right) \tag{4.1.6}$$

## 4.2 Lévy processes.

We now give the definition of Lévy processes. We restrict ourself to one-dimensional processes, the multidimensional ones are treated in [2] and [64]. We preliminarly recall that a process is continous in probability (or stochastically continuous) if $\lim_{h\to 0} X(t+h) \stackrel{p}{=} X(t)$, namely $\lim_{h\to 0} \Pr\{|X(t+h) - X(t)| > \epsilon\} = 0$ for each $\epsilon > 0$.

**Definition 3.** A real valued process $X = \{X(t), t > 0\}$ is said to be a Lévy process if

1. X(0)=0 almost surely
2. X has independent and stationary increments
3. X is continuous in probability

A fundamental consequence of the previous definition is the following:

**Theorem 4.** *If $X$ is a Lévy process, then, for each $t > 0$ the random variable $X(t)$ is infinitely divisible and its characteristic function is*

$$\phi_t(u) = \mathbb{E}e^{iuX(t)} = e^{t\eta(u)} \tag{4.2.1}$$

*where*

$$\eta(u) = ibu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}-\{0\}} (e^{iuy} - 1 - iuy\,\chi_{[-1,1](y)})\nu(dy). \tag{4.2.2}$$

*Proof.* Since the increments are independent and stationary, it follows that

$$\phi_{t+s}(u) = \phi_t(u)\phi_s(u) \tag{4.2.3}$$

which gives, together with the initial condition $\phi_0(u) = 1$, the unique solution $e^{t\eta(u)}$ for a suitable constant $\eta(u)$ independent on $t$. We have to calculate $\eta(u)$.

For each $n \in \mathbb{N}$, it is sufficient to write

$$X(t) = \sum_{k=0}^{n} X\left(\frac{kt}{n}\right) - X\left(\frac{(k-1)t}{n}\right) = \sum_{k=0}^{n} Y_k \tag{4.2.4}$$

and observe that, since independence and stationareity of the increments, the variables $Y_k$ are i.i.d, and this proves the infinite divisibility of $X(t)$.

Observe now that

$$\phi_t(u) = (\phi_1(u))^t \tag{4.2.5}$$

Since $X(1)$ infinitely divisible, we can apply the Lévy -Khintchine formula (4.1.6) to compute $\phi_1(u)$ and this concludes the proof. $\square$

We conclude this section by explaining the construction of the sample paths of Lévy processes. The following theorem, known as Lévy-Ito decomposition, states that any Lévy process is given by the sum of three independent processes. The first is a a diffusion process (the brownian motion with drift is indeed the unique continuous path Lévy process), the second is a Compound poisson process whose jumps are "sufficiently large", the third is the limit of a sequence of Compound Poisson processes with "small" jumps, compensated by its mean. In some sense, this is the path-space version of the Lévy - khinchine formula, which only regarded with the univariate probability distribution of $X(t)$.

**Theorem 5.** *For any function of the form (4.1.6), there exists a Lévy process $X$ whose univariate distribution has characteristic function (4.1.6). Moreover $X$ is the sum of three processes:*

$$X(t) = X_1(t) + X_2(t) + X_3(t) \qquad a.s. \tag{4.2.6}$$

*where $X_1(t)$ has Lévy exponent*

$$\phi_1(u) = ibu - \frac{1}{2}\sigma^2 u^2 \tag{4.2.7}$$

*, $X_2$ is a compound Poisson process with Lévy exponent*

$$\phi_2(u) = \int_{|y|>1} (e^{iuy} - 1)\nu(dy) \tag{4.2.8}$$

*and $X_3$ is the limit as $n \to \infty$ of compensated Poisson processes with Lévy exponents*

$$\phi_3(u) = \int_{\frac{1}{n}<|y|<1} (e^{iuy} - 1)\nu(dy) - iu \int_{\frac{1}{n}<|y|<1} y\,\nu(dy). \tag{4.2.9}$$

## 4.3 Subordinators.

An important class of one-dimensional Lévy processes is formed by the so-called subordinators.

**Definition 4.** A Lévy process $\{H(t), t > 0\}$ is a subordinator if it is increasing almost surely and its Lévy measure satisfies the additional condition

$$\int_0^\infty (x \wedge 1)\nu(dx) < \infty. \tag{4.3.1}$$

.

Of course, a Levy process with characteristic function defined by the triplet $(b, \sigma^2, \nu)$, is a subordinator if and only if $b \geq 0$, $\sigma^2 = 0$ (i.e. the Brownian part must be absent since its trajectories are not increasing) and $\nu(-\infty, 0) = 0$ (namely only upward jumps occurr).

Since a subordinator has positive values, its Laplace transform is well-defined and is given in the following statement.

**Proposition 2.** *Let $\{H(t), t > 0\}$ be a subordinator. Then there exists a constant $\beta \geq 0$ and a measure $\nu$ on $(0, \infty)$ satisfying 4.3.1 such that $\mathbb{E}e^{-\lambda H(t)} = e^{-tf(\lambda)}$, where*

$$f(\lambda) = \beta\lambda + \int_0^\infty (1 - e^{-\lambda x})\nu(dx). \tag{4.3.2}$$

*Proof.* By using (4.2.2), with the positions $\sigma^2 = 0$ and $\nu(-\infty, 0) = 0$, the Laplace transform is obtained by putting $u = i\lambda$

$$\mathbb{E}e^{-\lambda X(t)} = e^{-tf(\lambda)} \tag{4.3.3}$$

where

$$f(\lambda) = b\lambda + \int_{\mathbb{R}^+} \big(1 - e^{-\lambda y} - \lambda\, y\chi_{[0,1]}(y)\big).\nu(dy) \tag{4.3.4}$$

Thanks to 4.3.1 we can split the integral in the following way

$$f(\lambda) = b\lambda + \int_{\mathbb{R}^+} (1 - e^{-\lambda y})\nu(dy) - \lambda \int_0^1 y\,\nu(dy) \tag{4.3.5}$$

and finally incorporate the new term in the drift

$$\beta = b - \int_0^1 y\nu(dy) \tag{4.3.6}$$

and the proof is complete. □

The previous result also suggests that a subordinator can be constructed as the sum of a drift and a (not-compensated) compound Poisson process

$$H(t) = bt + \sum_{0\leq s\leq t} \Delta H(t) \tag{4.3.7}$$

where $\Delta H(t) = H(t) - H(t^-)$. Note that (4.3.7) is the Lévy -Ito decomposition (4.3.7) adapted to subordinators.

The following result shows that the probability distribution of any subordinator can be approximated by means of that of Compound Poisson Processes.

**Theorem 6.** *Let $H$ be a subordinator. Then there exists a family of Compound Poisson Processes $Z_\gamma$, such that $Z_\gamma(t)$ converges in distribution to $H(t)$ as $\gamma \to 0$.*

*Proof.* Let $\nu(dx)$ be the Lévy measure of a subordinator. In general, it is not integrable as $x \to 0$, as clear from (4.3.1). However, by considering the tail of the measure,

$$\overline{\nu}(\gamma) = \int_\gamma^\infty \nu(dx) \qquad \gamma > 0 \tag{4.3.8}$$

it is obvious that

$$\psi_\gamma(dx) = \frac{\nu(dx)}{\overline{\nu}(\gamma)} \qquad x \geq \gamma \tag{4.3.9}$$

is a probability measure, since it is positive and integrates to 1. The Compund Poisson Process

$$Z_\gamma(t) = \sum_{j=0}^{N_\gamma(t)} X_j^\gamma, \tag{4.3.10}$$

where $N_\gamma(t)$ is a Poisson process with rate $\frac{1}{\overline{\nu}(\gamma)}$ and each $X_j^\gamma$ has distribution $\psi_\gamma(dx)$, has Laplace transform

$$\mathbb{E}e^{-uZ_\gamma(t)} = e^{-t\int_\gamma^\infty(1-e^{-ux})\nu(dx)} \tag{4.3.11}$$

which clearly converges to the Laplace transform of $H(t)$ as $\gamma \to 0$. □

To extend the class of subordinators, we introduce the so-called "killed" subordinators: at a random time, which is stochastically independent on the process itself, a killed subordinator makes a jump of infinite size and reaches the absorbing state (or cimitery point) at $\infty$. A killed subordinator is constructed as

$$\tilde{H}(t) = \begin{cases} H(t) & t < T \\ \infty & t \geq T \end{cases} \tag{4.3.12}$$

where $H(t)$ is a subordinator in the strict sense and $T$ is the random life-time of the process, having probability distribution $Exp(a)$.

The Laplace tranform of 4.3.12 is

$$\mathbb{E}e^{-\lambda\tilde{H}(t)} = \mathbb{E}e^{-\lambda H(t)}\Pr\{t < T\} = e^{-tf(\lambda)}e^{-at} \tag{4.3.13}$$

and the Laplace exponent is

$$g(\lambda) = a + f(\lambda) = a + b\lambda + \int_0^\infty (1-e^{-\lambda x})\nu(dx) \tag{4.3.14}$$

**Remark 1.** *It is now appropriate to do an excursus of mathematical analysis. A non-negative function $g \in C^\infty(0,\infty)$ is said to be a Bernstein function if $(-1)^n g^{(n)} \leq 0$ for $n \in \mathbb{N}$. It is possible to prove that each Bernstein function has the form*

$$g(\lambda) = a + b\lambda + \int_0^\infty (1-e^{-\lambda x})\nu(dx) \tag{4.3.15}$$

*for suitable $a > 0$, $b > 0$ and a measure $\nu$ satisfying (4.3.1). Hence, the class of all the Laplace exponents of subordinators coincides with the class of all Bernstein functions.*

# Chapter 5

# Subordinated processes.

## 5.1 Introduction

In this chapter, we consider time-changed models of population evolution $\mathcal{X}^f(t) = \mathcal{X}(H^f(t))$, where $\mathcal{X}$ is a counting process and $H^f$ is a subordinator with Laplace exponent $f$. In the case $\mathcal{X}$ is a pure birth process, we study the form of the distribution, the intertimes between successive jumps and the condition of explosion (also in the case of killed subordinators). We also investigate the case where $\mathcal{X}$ represents a death process (linear or sublinear) and study the extinction probabilities as a function of the initial population size $n_0$. Finally, the subordinated linear birth-death process is considered. A special attention is devoted to the case where birth and death rates coincide; the sojourn times are also analysed.

Birth and death processes can be applied in modelling many dynamical systems, such as cosmic showers, fragmentation processes, queueing systems, epidemics, population growth and aftershocks in earthquakes. The time-changed version of such processes has also been analysed since it is useful to describe the dynamics of various systems when the underlying environmental conditions randomly change. For example, the fractional birth and death processes, studied in Orsingher and Polito [54, 55, 56]; Orsingher, Ricciuti, Toaldo [59], are time-changed processes where the distribution of the time is related to the fractional diffusion equations. On this point consult Cahoy and Polito [15, 16] for some applications and simulations.

In this chapter, we consider the case where the random time is a subordinator. Actually, subordinated Markov processes have been extensively studied since the Fifties. The case of birth and death processes merits however a further investigation and this is the role of the present study. We consider here compositions of point processes $\mathcal{X}(t)$, $t > 0$, with an arbitrary subordinator $H^f(t)$ related to the Bernštein functions $f$. We denote such processes as $\mathcal{X}^f(t) = \mathcal{X}(H^f(t))$. The general form of $f$ is as follows

$$f(x) = \alpha + \beta x + \int_0^\infty (1 - e^{-xs})\nu(ds) \qquad \alpha \geq 0, \beta \geq 0, \tag{5.1.1}$$

where $\nu$ is the Lévy measure satisfying

$$\int_0^\infty (s \wedge 1)\nu(ds) < \infty. \tag{5.1.2}$$

In this chapter we refer to the case $\alpha = \beta = 0$, unless explicitly stated. The structure of the chapter is as follows: section 2 treats the subordinated non-linear birth process; section 3 deals with the subordinated linear and sublinear death processes; section 4 analyses the linear birth-death process, with particular attention to the case where birth and death rates coincide. In all three cases, we compute directly the state probabilities by means of the composition formula

$$\Pr\left\{\mathcal{X}^f(t) = k\right\} = \int_0^\infty \Pr\left\{\mathcal{X}(s) = k\right\} \Pr\left\{H^f(t) \in ds\right\}. \tag{5.1.3}$$

Despite most of the subordinators do not possess an explicit form for the probability density function, the distribution of $\mathcal{X}(H^f(t))$ always presents a closed form in terms of the Laplace exponent $f$. We also study the transition probabilities, both for finite and infinitesimal time intervals. We emphasize that the subordinated point processes have a fundamental difference with respect to the classical ones, in that they perform upward or downward jumps of arbitrary size. For infinitesimal time intervals, we provide a direct and simple proof of the following fact:

$$\Pr\left\{\mathcal{X}^f(t+dt) = k | \mathcal{X}^f(t) = r\right\} = dt \int_0^\infty \Pr\left\{\mathcal{X}(s) = k | \mathcal{X}(0) = r\right\} \nu(ds), \tag{5.1.4}$$

which is related to Bochner subordination (see [62]).
The first case taken into account is that of a non-linear birth process with birth rates $\lambda_k$, $k \geq 1$, which is denoted by $\mathcal{N}(t)$. The subordinated process $\mathcal{N}^f(t)$ does not explode if and only if the following condition is fullfilled

$$\sum_{j=1}^\infty \frac{1}{\lambda_j} = \infty. \tag{5.1.5}$$

This is the same condition of non-explosion holding for the classical case. Such a condition ceases to be true if we consider a Lévy exponent with $\alpha \neq 0$, which is related to the so-called killed subordinator. In this case, indeed, the process $\mathcal{N}^f(t)$ can explode in a finite time, even if $\mathcal{N}(t)$ does not; more precisely

$$\Pr\left\{\mathcal{N}^f(t) = \infty\right\} = 1 - e^{-\alpha t}. \tag{5.1.6}$$

We note that $\mathcal{N}^f(t)$ can be regarded as a process where upward jumps are separated by exponentially distribuited time intervals $Y_k$ such that

$$\Pr\left\{Y_k > t | \mathcal{N}^f(T_{k-1}) = r\right\} = e^{-f(\lambda_r)t} \tag{5.1.7}$$

where $T_{k-1}$ is the instant of the $(k-1)$-th jump.

In section 3 we study the subordinated linear and sublinear death processes, that we respectively denote by $M^f(t)$ and $\mathbb{M}^f(t)$, with an initial number of components $n_0$. We emphasize that in the sublinear case the annihilation is initially slower, then accelerates when few survivors remain. So, despite $M^f(t)$ and $\mathbb{M}^f(t)$ present different state probabilities, we observe that the extinction probabilities coincide and we prove that they decrease for increasing values of $n_0$.

In section 4, the subordinated linear birth-death process $L^f(t)$ is considered. If the birth and death rates coincide and $H^f$ is a stable subordinator, we compute the mean sojourn time in each state and find, in some particular cases, the distribution of the intertimes between successive jumps. We finally study the probability density of the sojourn times, by giving a sketch of the derivation of their Laplace transforms.

## 5.2 Subordinated non-linear birth process

We consider in this section the process $\mathcal{N}^f(t) = \mathcal{N}(H^f(t))$, where $\mathcal{N}$ is a non-linear birth process with one progenitor and rates $\lambda_k$, $k \geq 1$, and $H^f(t)$ is a subordinator independent from $\mathcal{N}(t)$. It is well known that the state probabilities of $\mathcal{N}(t)$ read

$$\Pr\left\{\mathcal{N}(t) = k | \mathcal{N}(0) = 1\right\} = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{e^{-\lambda_m t}}{\prod_{l=1, l\neq m}^{k}(\lambda_l - \lambda_m)}, & k > 1, \\ e^{-tf(\lambda_1)}, & k = 1. \end{cases} \tag{5.2.1}$$

The subordinated process $\mathcal{N}^f(t)$ thus possesses the following distribution:

$$\begin{aligned} \Pr\left\{\mathcal{N}^f(t) = k | \mathcal{N}^f(0) = 1\right\} &= \int_0^\infty \Pr\left\{\mathcal{N}(s) = k | \mathcal{N}(0) = 1\right\} \Pr\left\{H^f(t) \in ds\right\} \\ &= \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{e^{-t\, f(\lambda_m)}}{\prod_{l=1, l\neq m}^{k}(\lambda_l - \lambda_m)}, & k > 1, \\ e^{-tf(\lambda_1)}, & k = 1. \end{cases} \end{aligned} \tag{5.2.2}$$

The distribution (5.2.2) can be easily generalised to the case of $r$ progenitors and reads

$$\Pr\left\{\mathcal{N}^f(t) = r + k | \mathcal{N}^f(0) = r\right\} = \begin{cases} \prod_{j=r}^{r+k-1} \lambda_j \sum_{m=r}^{r+k} \frac{e^{-tf(\lambda_m)}}{\prod_{l=r, l\neq m}^{r+k}(\lambda_l - \lambda_m)}, & k > 0, \\ e^{-tf(\lambda_r)}, & k = 0. \end{cases} \tag{5.2.3}$$

The subordinated process $\mathcal{N}^f(t)$ is time-homogeneous and Markovian. So, the last formula permits us to write

$$\begin{aligned} &\Pr\left\{\mathcal{N}^f(t+dt)=r+k|\mathcal{N}^f(t)=r\right\} \\ &= \begin{cases} \prod_{j=r}^{r+k-1}\lambda_j \sum_{m=r}^{r+k} \frac{1-dtf(\lambda_m)}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l-\lambda_m)}, & k>0, \\ 1-dtf(\lambda_r), & k=0. \end{cases} \end{aligned} \tag{5.2.4}$$

To find an alternative expression for the transition probabilities we need the following

**Lemma 5.2.1.** *For any sequence of $k+1$ distinct positive numbers $\lambda_r, \lambda_{r+1}\cdots\lambda_{r+k}$ the following relationship holds:*

$$c_{r,k}=\sum_{m=r}^{r+k}\frac{1}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l-\lambda_m)}=0. \tag{5.2.5}$$

*Proof.* It is a consequence of (5.2.3) by letting $t\to 0$. An alternative proof can be obtained by suitably adapting the calculation in Theorem 2.1 of [55]. □

We are now able to state the following theorem.

**Theorem 7.** *For $k>r$ the transition probability takes the form*

$$\Pr\left\{\mathcal{N}^f(t+dt)=k|\mathcal{N}^f(t)=r\right\}=dt\int_0^\infty \Pr\left\{\mathcal{N}(s)=k|\mathcal{N}(0)=r\right\}\nu(ds) \tag{5.2.6}$$

*Proof.* By repeatedly using both (5.2.5) and the representation (5.1.1) of the Bernštein functions $f$, we have that

$$\begin{aligned} \Pr\left\{\mathcal{N}^f(t+dt)=k|\mathcal{N}^f(t)=r\right\} &= \prod_{j=r}^{r+k-1}\lambda_j\sum_{m=r}^{r+k}\frac{1-dtf(\lambda_m)}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l-\lambda_m)} \\ &= -dt\prod_{j=r}^{r+k-1}\lambda_j\sum_{m=r}^{r+k}\frac{f(\lambda_m)}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l-\lambda_m)} \\ &= -dt\int_0^\infty\prod_{j=r}^{r+k-1}\lambda_j\sum_{m=r}^{r+k}\frac{1-e^{-\lambda_m s}}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l-\lambda_m)}\nu(ds) \\ &= dt\int_0^\infty\prod_{j=r}^{r+k-1}\lambda_j\sum_{m=r}^{r+k}\frac{e^{-\lambda_m s}}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l-\lambda_m)}\nu(ds). \end{aligned} \tag{5.2.7}$$

In light of (5.2.5), the integrand in (5.2.7) is $\mathcal{O}(s)$ for $s\to 0$. Reminding (5.1.2), this ensures the convergence of (5.2.7), and the proof is thus complete. □

**Remark 2.** *For the sake of completeness, we observe that in the case $k = 0$ we have*

$$\begin{aligned}\Pr\left\{\mathcal{N}^f(t+dt) = r|\mathcal{N}^f(t) = r\right\} &= 1 - dt f(\lambda_r) \\ &= 1 - dt\int_0^\infty (1 - e^{-\lambda_r s})\nu(ds) \qquad (5.2.8) \\ &= 1 - dt\int_0^\infty (1 - \Pr\{\mathcal{N}(s) = r|\mathcal{N}(0) = r\})\nu(ds).\end{aligned} \tag{5.2.9}$$

**Remark 3.** *The subordinated non-linear birth process performs jumps of arbitrary height as the subordinated Poisson process (see, for example, Orsingher and Toaldo [61]). Thus, in view of markovianity, we can write the governing equations for the state probabilities $p_k^f(t) = \Pr\left\{\mathcal{N}^f(t) = k|\mathcal{N}^f(0) = 1\right\}$. For $k > 1$ we have that*

$$\frac{d}{dt}p_k^f(t) = -f(\lambda_k)p_k^f(t) + \sum_{r=1}^{k-1} p_r^f(t)\int_0^\infty \prod_{j=r}^{k-1}\lambda_j \sum_{m=r}^{k} \frac{e^{-\lambda_m s}}{\prod_{l=r,l\neq m}^{k}(\lambda_l - \lambda_m)}\nu(ds), \tag{5.2.10}$$

*while for $k = 1$*

$$\frac{d}{dt}p_1^f(t) = -f(\lambda_1)p_1^f(t). \tag{5.2.11}$$

**Remark 4.** *The process $\mathcal{N}(H^f(t))$ presents positive and integer-valued jumps occurring at random times $T_1, T_2, \cdots T_n$. The inter-arrival times $Y_1, Y_2, \cdots Y_n$ are defined as*

$$Y_k = T_k - T_{k-1}. \tag{5.2.12}$$

*It is easy to prove that*

$$\Pr\left\{Y_k > t|\mathcal{N}^f(T_{k-1}) = r\right\} = e^{-f(\lambda_r)t}. \tag{5.2.13}$$

*This can be justified by considering that in the time intervals $[T_{k-1}, T_{k-1}+t]$, no new offspring appears in the population and thus, by (5.2.4), we have*

$$\Pr\left\{Y_k > t|\mathcal{N}^f(T_{k-1}) = r\right\} = \Pr\left\{\mathcal{N}^f(t+T_{k-1}) = r|\mathcal{N}^f(T_{k-1}) = r)\right\} = e^{-f(\lambda_r)t}. \tag{5.2.14}$$

### 5.2.1 Condition of explosion for the subordinated non-linear birth process

We note that the explosion of the process $\mathcal{N}^f(t)$, $t > 0$, in a finite time is avoided if and only if

$$T_\infty = Y_1 + Y_2 \cdots Y_\infty = \infty \tag{5.2.15}$$

where $Y_j$ , $j \geq 1$, are the intertimes between successive jumps (see [29], p. 252). For the non-linear classical process we have that

$$\begin{aligned} \mathbb{E}e^{-T_\infty} &= \mathbb{E}e^{-\sum_{j=1}^{\infty} Y_j} = \lim_{n\to\infty} \prod_{j=1}^{n} \mathbb{E}e^{-Y_j} = \lim_{n\to\infty} \prod_{j=1}^{n} \frac{\lambda_j}{1+\lambda_j} \\ &= \prod_{j=1}^{\infty} \frac{1}{1+\frac{1}{\lambda_j}} = \frac{1}{1+\sum_{j=1}^{\infty}\frac{1}{\lambda_j}+\cdots}. \end{aligned} \tag{5.2.16}$$

So, if $\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty$ we have $e^{-T_\infty} = 0$ a.s., that is $T_\infty = \infty$. Therefore, for the subordinated non-linear birth process we have that

$$\begin{aligned} \Pr\left\{\mathcal{N}^f(t) < \infty\right\} &= \int_0^\infty \sum_{k=1}^{\infty} \Pr\left\{\mathcal{N}(s) = k\right\} \Pr\left\{H^f(t) \in ds\right\} \\ &= \int_0^\infty \Pr\left\{H^f(t) \in ds\right\} = 1, \qquad \forall t > 0. \end{aligned} \tag{5.2.17}$$

Instead, if $\sum_{j=1}^{\infty} \frac{1}{\lambda_j} < \infty$, we get $\sum_{k=1}^{\infty} \Pr\left\{\mathcal{N}(s) = k\right\} < \infty$, and this implies that $\Pr\left\{\mathcal{N}^f(t) < \infty\right\} < 1$.

We can now consider the case of killed subordinators $\mathcal{H}^g(t)$, defined as

$$\mathcal{H}^g(t) = \begin{cases} H^f(t), & t < T, \\ \infty, & t \geq T, \end{cases} \tag{5.2.18}$$

where $T \sim Exp(\alpha)$ and $H^f(t)$ is an ordinary subordinator related to the function $f(x) = \int_0^\infty (1 - e^{-sx})\nu(ds)$. It is well-known that $\mathcal{H}^g(t)$ is related to a Bernštein function

$$g(x) = \alpha + f(x). \tag{5.2.19}$$

In this case, even if $\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty$ , the probability of explosion for $\mathcal{N}^f(t)$ is positive and equal to

$$\Pr\left\{\mathcal{N}^f(t) = \infty\right\} = 1 - e^{-t\alpha}. \tag{5.2.20}$$

This can be proven by observing that

$$\begin{aligned} \Pr\left\{\mathcal{N}^f(t) < \infty\right\} &= \int_0^\infty \sum_{k=1}^{\infty} \Pr\left\{\mathcal{N}(s) = k\right\} \Pr\left\{H^f(t) \in ds\right\} \\ &= \int_0^\infty \Pr\left\{H^f(t) \in ds\right\} = \int_0^\infty e^{-\mu s} \Pr\left\{H^f(t) \in ds\right\} \Big|_{\mu=0} \\ &= e^{-\alpha t - f(\mu) t}\Big|_{\mu=0} = e^{-\alpha t}. \end{aligned} \tag{5.2.21}$$

If, instead, $\sum_{j=1}^{\infty} \frac{1}{\lambda_j} < \infty$, we have $\sum_{k=1}^{\infty} \Pr\left\{\mathcal{N}(s) = k\right\} < 1$ and, a fortiori, $\Pr\left\{\mathcal{N}^f(t) < \infty\right\} < e^{-\alpha t}$.

### 5.2.2 Subordinated linear birth process

The subordinated Yule-Furry process $N^f(t)$ with one initial progenitor possesses the following distribution

$$\begin{aligned} p_k^f(t) &= \int_0^\infty e^{-\lambda s}(1-e^{-\lambda s})^{k-1} \Pr\{H^f(t) \in ds\} \\ &= \int_0^\infty e^{-\lambda s} \sum_{J=0}^{k-1} \binom{k-1}{j} (-1)^j e^{-\lambda s j} \Pr\{H^f(t) \in ds\} \\ &= \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \int_0^\infty e^{-s(\lambda+\lambda j)} \Pr\{H^f(t) \in ds\} \\ &= \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j e^{-t\, f(\lambda(j+1))}. \end{aligned} \tag{5.2.22}$$

Of course, this is obtainable from the distribution $\mathcal{N}^f(t)$ by assuming that $\lambda_j = \lambda j$ We now compute the factorial moments of the subordinated linear birth process. The probability generating function is

$$G^f(u,t) = \sum_{k=1}^\infty u^k \int_0^\infty e^{-\lambda s}(1-e^{-\lambda s})^{k-1} \Pr(H^f(t) \in ds). \tag{5.2.23}$$

The $r$-th order factorial moments are

$$\begin{aligned} &\frac{\partial^r}{\partial u^r} G^f(u,t)\bigg|_{u=1} \\ &= \sum_{k=r}^\infty k(k-1)\cdots(k-r+1) \int_0^\infty e^{-\lambda s}(1-e^{-\lambda s})^{k-1} \Pr\left\{H^f(t) \in ds\right\} \\ &= \sum_{k=r}^\infty k(k-1)\cdots(k-r+1) \int_0^\infty e^{-\lambda s}(1-e^{-\lambda s})^{k-r}(1-e^{-\lambda s})^{r-1} \Pr\left\{H^f(t) \in ds\right\} \end{aligned} \tag{5.2.24}$$

and since

$$\sum_{k=r}^\infty k(k-1)...(k-r+1)(1-p)^{k-r} = (-1)^r \frac{d^r}{dp^r} \sum_{k=0}^\infty (1-p)^k = (-1)^r \frac{d^r}{dp^r}\frac{1}{p} = \frac{r!}{p^{r+1}} \tag{5.2.25}$$

we have that

$$\begin{aligned} \frac{\partial^r}{\partial u^r} G(u,t)\bigg|_{u=1} &= r! \int_0^\infty e^{\lambda r s}(1-e^{-\lambda s})^{r-1} \Pr\left\{H^f(t) \in ds\right\} \\ &= r! \sum_{m=0}^{r-1} \binom{r-1}{m} (-1)^m \int_0^\infty e^{-\lambda s(m-r)} \Pr\left\{H^f(t) \in ds\right\} \end{aligned} \tag{5.2.26}$$

$$= r! \sum_{m=0}^{r-1} \binom{r-1}{m} (-1)^m e^{-tf(\lambda(m-r))}. \tag{5.2.27}$$

By $f(-x)$, $x > 0$ we mean the extended Bernštein function, having representation

$$f(-x) = \int_0^\infty (1 - e^{sx}) \nu(ds), \qquad x > 0, \tag{5.2.28}$$

provided that the integral in (5.2.28) is convergent. In particular, we infer that

$$\mathbb{E}(\mathcal{N}^f(t)) = e^{-tf(-\lambda)} \tag{5.2.29}$$

and

$$\mathrm{Var}(\mathcal{N}^f(t)) = 2e^{-tf(-2\lambda)} - e^{-tf(-\lambda)} - e^{-2tf(-\lambda)}. \tag{5.2.30}$$

For a stable subordinator, that is with Lévy measure $\nu(ds) = \frac{\alpha s^{-\alpha-1}}{\Gamma(1-\alpha)} ds$ , $\alpha \in (0,1)$, all the factorial moments are infinite. Instead, for a tempered stable subordinator, where $\nu(ds) = \frac{\alpha e^{-\theta s} s^{-\alpha-1}}{\Gamma(1-\alpha)} ds$, $\alpha \in (0,1)$ and $\theta > 0$, only the factorial moments of order $r$ such that $r < \frac{\theta}{\lambda}$ are finite. If we then consider the Gamma subordinator, with $\nu(ds) = \frac{e^{-\alpha s}}{s} ds$, only the factorial moments of order $r$ such that $r < \frac{\alpha}{\lambda}$ are finite.

### 5.2.3 Fractional subordinated non-linear birth process

The fractional non-linear birth process has state probabilities $p_k^\nu(t)$ solving the fractional differential equation

$$\frac{d^\nu p_k^\nu(t)}{dt^\nu} = -\lambda_k p_k^\nu(t) + \lambda_{k-1} p_{k-1}^\nu(t) \qquad \nu \in (0,1), k \geq 1 \tag{5.2.31}$$

with initial condition

$$p_k^\nu(0) = \begin{cases} 1, & k = 1, \\ 0, & k > 1. \end{cases} \tag{5.2.32}$$

The state probabilities read (see Orsingher and Polito [56])

$$p_k^\nu(t) = \Pr\{\mathcal{N}^\nu(t) = k | \mathcal{N}^\nu(0) = 1\} = \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{E_{\nu,1}(-\lambda_m t^\nu)}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)} \qquad \nu \in (0,1), \tag{5.2.33}$$

where

$$E_{\nu,1}(-\eta t^\nu) = \frac{\sin(\nu\pi)}{\pi} \int_0^\infty \frac{r^{\nu-1} e^{-r\eta^{\frac{1}{\nu}} t}}{r^{2\nu} + 2r^\nu \cos(\nu\pi) + 1} dr \tag{5.2.34}$$

is the Mittag-Leffler function (see formula (7.3) in Haubold et al. [30]). So, the subordinated non-linear fractional birth process has distribution

$$\Pr\left\{\mathcal{N}^\nu(H^f(t)) = k|\mathcal{N}^\nu(0)=1\right\}$$
$$= \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} \frac{\sin(\nu\pi)}{\pi}\int_0^\infty \frac{r^{\nu-1}e^{-tf(r\lambda_m^{\frac{1}{\nu}})}}{r^{2\nu}+2r^\nu\cos(\nu\pi)+1}dr. \tag{5.2.35}$$

## 5.3 Subordinated death processes

We now consider the process $M^f(t) = M(H^f(t))$, where $M$ is a linear death process with $n_0$ progenitors. The state probabilities read

$$\Pr\left\{M^f(t)=k|M^f(0)=n_0\right\} = \int_0^\infty \binom{n_0}{k} e^{-\mu ks}(1-e^{-\mu s})^{n_0-k}\Pr\left\{H^f(t)\in ds\right\}$$
$$= \binom{n_0}{k}\sum_{j=0}^{n_0-k}\binom{n_0-k}{j}(-1)^j\int_0^\infty e^{-(\mu k+\mu j)s}\Pr\left\{H^f(t)\in ds\right\}$$
$$= \binom{n_0}{k}\sum_{j=0}^{n_0-k}\binom{n_0-k}{j}(-1)^j e^{-tf(\mu k+\mu j)}. \tag{5.3.1}$$

In particular, the extinction probability is

$$\begin{aligned}\Pr\left\{M^f(t)=0|M^f(0)=n_0\right\} &= \sum_{j=0}^{n_0}\binom{n_0}{j}(-1)^j e^{-tf(\mu j)}\\ &= 1+\sum_{j=1}^{n_0}\binom{n_0}{j}(-1)^j e^{-tf(\mu j)}\end{aligned} \tag{5.3.2}$$

and converges to 1 exponentially fast with rate $f(\mu)$.

**Remark 5.** *We observe that the extinction probability is a decreasing function of $n_0$ for any choice of the subordinator $H^f(t)$. This can be shown by observing that*

$$\Pr\left\{M^f(t)=0|M^f(0)=n_0\right\} - \Pr\left\{M^f(t)=0|M^f(0)=n_0-1\right\}$$
$$= \sum_{j=1}^{n_0}\binom{n_0}{j}(-1)^j e^{-tf(\mu j)} - \sum_{j=1}^{n_0-1}\binom{n_0-1}{j}(-1)^j e^{-tf(\mu j)}$$
$$= \sum_{j=1}^{n_0-1}\binom{n_0-1}{j-1}(-1)^j e^{-tf(\mu j)} + (-1)^{n_0}e^{-tf(\mu n_0)}$$
$$= \sum_{j=1}^{n_0}\binom{n_0-1}{j-1}(-1)^j e^{-tf(\mu j)}$$

$$
\begin{aligned}
&= -\sum_{j=0}^{n_0-1}\binom{n_0-1}{j}(-1)^j e^{-tf(\mu(j+1)} \\
&= -\int_0^\infty \sum_{j=0}^{n_0-1}\binom{n_0-1}{j}(-1)^j e^{-s\mu(j+1)} \Pr\left\{H^f(t)\in ds\right\} \\
&= -\int_0^\infty e^{-\mu s}(1-e^{-\mu s})^{n_0-1}\Pr\left\{H^f(t)\in ds\right\} < 0. \qquad (5.3.3)
\end{aligned}
$$

*This permits us also to establish the following upper bound which is valid for all values of $n_0$.*

$$
\Pr\left\{M^f(t)=0|M^f(0)=n_0\right\} < \Pr\left\{M^f(t)=0|M^f(0)=1\right\} = 1-e^{-tf(\mu)}. \qquad (5.3.4)
$$

*We also infer that*

$$
\begin{aligned}
&\Pr\left\{M^f(t)=k|M^f(0)=n_0\right\} = \\
&\Pr\left\{M^f(t)=k|M^f(0)=n_0-1\right\} - \frac{1}{n_0}\Pr\left\{M^f(t)=1|M^f(0)=n_0\right\} \qquad \forall k<n_0
\end{aligned}
$$

**Remark 6.** *The probability generating function of the subordinated linear death process is*

$$
G(u,t) = \int_0^\infty (ue^{-\mu s}+1-e^{-\mu s})^{n_0}\Pr\left\{H^f(t)\in ds\right\}. \qquad (5.3.5)
$$

*We now compute the factorial moments of order $r$ for the process $M^f(t)$:*

$$
\begin{aligned}
&\mathbb{E}\big(M^f(t)(M^f(t)-1)(M^f(t)-2)\cdots(M^f(t)-r+1)\big) \\
&= \int_0^\infty \frac{\partial^r}{\partial u^r}(ue^{-\mu s}+1-e^{-\mu s})^{n_0}|_{u=1}\Pr\left\{H^f(t)\in ds\right\} \\
&= n_0(n_0-1)(n_0-2)...(n_0-r+1)\int_0^\infty e^{-\mu r s}\Pr\left\{H^f(t)\in ds\right\} \\
&= n_0(n_0-1)(n_0-2)...(n_0-r+1)e^{-tf(\mu r)} \\
&= r!\binom{n_0}{r}e^{-tf(\mu r)} \qquad \textit{for } r\le n_0. \qquad (5.3.6)
\end{aligned}
$$

*In particular, we extract the expressions*

$$
\mathbb{E}\, M^f(t) = n_0 e^{-t\,f(\mu)} \qquad (5.3.7)
$$

*and*

$$
Var\, M^f(t) = n_0e^{-tf(\mu)} - n_0e^{-tf(2\mu)} + n_0^2e^{-tf(2\mu)} - n_0^2e^{-2tf(\mu)}. \qquad (5.3.8)
$$

*The variance can be also be obtained as*

$$
\begin{aligned}
Var\, M^f(t) &= \mathbb{E}\left\{ Var(M(H^f(t))|H^f(t))\right\} + Var\left\{\mathbb{E}(M(H^f(t))|H^f(t))\right\} \\
&= \mathbb{E}\big(n_0e^{-\mu H^f(t)}(1-e^{-\mu H^f(t)})\big) + Var(n_0e^{-\mu H^f(t)}) \\
&= n_0e^{-tf(\mu)} - n_0e^{-tf(2\mu)} + n_0^2e^{-tf(2\mu)} - n_0^2e^{-2tf(\mu)}. \qquad (5.3.9)
\end{aligned}
$$

**Remark 7.** *The transition probabilities*

$$\Pr\left\{M^f(t_0+t)=k|M^f(t_0)=r\right\}=\binom{r}{k}\sum_{j=0}^{r-k}\binom{r-k}{j}(-1)^j e^{-tf(\mu k+\mu j)} \tag{5.3.10}$$

*permit us to write, for a small time interval* $[t,t+dt)$,

$$\begin{aligned}
&\Pr\left\{M^f(t_0+dt)=k|M^f(t_0)=r\right\}\\
=&\binom{r}{k}\sum_{j=0}^{r-k}\binom{r-k}{j}(-1)^j(1-dt\,f(\mu k+\mu j))\\
=&-dt\binom{r}{k}\sum_{j=0}^{r-k}\binom{r-k}{j}(-1)^j\,f(\mu k+\mu j)\\
=&-dt\binom{r}{k}\sum_{j=0}^{r-k}\binom{r-k}{j}(-1)^j\int_0^\infty(1-e^{-(\mu k+\mu j)s})\nu(ds)\\
=&\,dt\binom{r}{k}\int_0^\infty\sum_{j=0}^{r-k}\binom{r-k}{j}(-1)^j\,e^{-\mu js}e^{-\mu ks}\nu(ds)\\
=&\,dt\int_0^\infty\binom{r}{k}(1-e^{-\mu s})^{r-k}e^{-\mu ks}\nu(ds)\\
=&\,dt\int_0^\infty\Pr\left\{M(s)=k|M(0)=r\right\}\nu(ds)\qquad 0\le k<r\le n_0
\end{aligned} \tag{5.3.11}$$

*It follows that the subordinated death process decreases with downwards jumps of arbitrary size. Formula* (5.3.11) *is a special case of* (5.1.4) *for the linear death process.*

**Remark 8.** *If* $M^f(t_0)=r$*, the probability that the number of individuals does not change during a time interval of length* $t$ *is*

$$\Pr\left\{M^f(t_0+t)=r|M^f(t_0)=r\right\}=e^{-tf(r\mu)}. \tag{5.3.12}$$

*As a consequence, the random time between two successive jumps has exponential distribution with rate* $f(\mu r)$*, i.e.*

$$T_r\sim Exp(f(\mu r)). \tag{5.3.13}$$

*From* (6.3.12) *we have also that*

$$\Pr\left\{M^f(t+dt)=r|M^f(t)=r\right\}=1-dtf(\mu r). \tag{5.3.14}$$

**Remark 9.** *In view of* (5.3.11) *we can write the governing equations for the transition probabilities* $p_k^f(t)=\Pr\left\{M^f(t)=k|M^f(0)=n_0\right\}$*, for* $0\le k\le n_0$

$$\frac{d}{dt}p_k^f(t)=-p_k^f(t)f(\mu k)+\sum_{j=k+1}^{n_0}p_j^f(t)\int_0^\infty\binom{j}{k}(1-e^{-\mu s})^{j-k}e^{-\mu ks}\nu(ds). \tag{5.3.15}$$

### 5.3.1 The subordinated sublinear death process

In the sublinear death process we have that, for $0 \le k \le n_0$,

$$\Pr\left\{\mathbb{M}(t+dt) = k-1 | \mathbb{M}(t) = k, \mathbb{M}(0) = n_0\right\} = \mu(n_0-k+1)dt + o(dt) \quad (5.3.16)$$

so that the probability that a particle disappears in $[t, t+dt)$ is proportional to the number of deaths occurred in $[0,t)$. It is well-known that

$$\Pr\left\{\mathbb{M}(t) = k | \mathbb{M}(0) = n_0\right\} = \begin{cases} e^{-\mu t}(1-e^{-\mu t})^{n_0-k}, & k = 1, 2, \dots, n_0, \\ (1-e^{-\mu t})^{n_0}, & k = 0. \end{cases} \quad (5.3.17)$$

So, the probability law of the subordinated process immediately follows

$$\begin{aligned} &\Pr\left\{\mathbb{M}^f(t) = k | \mathbb{M}^f(0) = n_0\right\} \\ &= \begin{cases} \sum_{j=0}^{n_0-k} \binom{n_0-k}{j} (-1)^j e^{-tf(\mu(j+1))}, & k = 0, 1, \dots, n_0, \\ \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k e^{-tf(\mu k)}, & k = 0 \end{cases} \end{aligned} \quad (5.3.18)$$

The extinction probability is a decreasing function of $n_0$ as in the sublinear death process. Furthermore we observe that the extinction probabilities for the subordinated linear and sublinear death process coincide.

## 5.4 Subordinated linear birth-death processes

In this section we consider the linear birth and death process $L(t)$ with one progenitor at the time $H^f(t)$. We recall that, for $k \ge 1$ (see Bailey [4], page 90),

$$\Pr\left\{L(t) = k | L(0) = 1\right\} = \begin{cases} \frac{(\lambda-\mu)^2 e^{-(\lambda-\mu)t}(\lambda(1-e^{-(\lambda-\mu)t}))^{k-1}}{(\lambda-\mu e^{-(\lambda-\mu)t})^{k+1}}, & \lambda > \mu, \\ \frac{(\mu-\lambda)^2 e^{-(\mu-\lambda)t}(\lambda(1-e^{-(\mu-\lambda)t}))^{k-1}}{(\mu-\lambda e^{-(\mu-\lambda)t})^{k+1}}, & \lambda < \mu, \\ \frac{(\lambda t)^{k-1}}{(1+\lambda t)^{k+1}}, & \lambda = \mu. \end{cases} \quad (5.4.1)$$

while the extinction probabilities have the form

$$\Pr\left\{L(t) = 0 | L(0) = 1\right\} = \begin{cases} \frac{\mu-\mu e^{-t(\lambda-\mu)}}{\lambda-\mu e^{-t(\lambda-\mu)}}, & \lambda > \mu, \\ \frac{\mu-\mu e^{-t(\mu-\lambda)}}{\lambda-\mu e^{-t(\mu-\lambda)}}, & \mu > \lambda, \\ \frac{\lambda t}{1+\lambda t}, & \lambda = \mu. \end{cases} \quad (5.4.2)$$

We now study the subordinated process $L^f(t) = L(H^f(t))$. When $\lambda \neq \mu$, after a series expansion we easily obtain that

$$\begin{aligned} &\Pr\left\{L^f(t) = k | L^f(0) = 1\right\} \\ &= \begin{cases} \left(\frac{\lambda-\mu}{\lambda}\right)^2 \sum_{l=0}^{\infty} \binom{l+k}{l} \left(\frac{\mu}{\lambda}\right)^l \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} e^{-tf((\lambda-\mu)(l+r+1))}, & \lambda > \mu, \\ \left(\frac{\mu-\lambda}{\mu}\right)^2 \left(\frac{\lambda}{\mu}\right)^{k-1} \sum_{l=0}^{\infty} \binom{l+k}{l} \left(\frac{\lambda}{\mu}\right)^l \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} e^{-tf((\mu-\lambda)(l+r+1))}, & \lambda < \mu, \end{cases} \end{aligned} \tag{5.4.3}$$

provided that $k \geq 1$. Moreover, the extinction probabilities have the following form

$$\Pr\left\{L^f(t) = 0\right\} = \begin{cases} \frac{\mu-\lambda}{\lambda} \left(\sum_{m=1}^{\infty} \left(\frac{\mu}{\lambda}\right)^m e^{-tf((\lambda-\mu)m)}\right) + \frac{\mu}{\lambda}, & \lambda > \mu, \\ 1 - \left(\frac{\mu-\lambda}{\lambda}\right) \sum_{m=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^m e^{-tf((\mu-\lambda)m)}, & \lambda < \mu. \end{cases} \tag{5.4.4}$$

Similarly to the classical process, we have

$$\lim_{t\to\infty} \Pr\left\{L^f(t) = 0\right\} = \begin{cases} \frac{\mu}{\lambda}, & \lambda > \mu, \\ 1, & \lambda < \mu. \end{cases} \tag{5.4.5}$$

### 5.4.1 Processes with equal birth and death rates

We concentrate ourselves on the case $\lambda = \mu$, which leads to some interesting results. The extinction probability reads

$$\begin{aligned} \Pr\left\{L^f(t) = 0 | L^f(0) = 1\right\} &= \int_0^\infty \frac{\lambda s}{1+\lambda s} \Pr\left\{H^f(t) \in ds\right\} \\ &= 1 - \int_0^\infty \frac{1}{1+\lambda s} \Pr\left\{H^f(t) \in ds\right\} \\ &= 1 - \int_0^\infty \Pr\left\{H^f(t) \in ds\right\} \int_0^\infty dw\, e^{-w\lambda s}\, e^{-w} \\ &= 1 - \int_0^\infty dw\, e^{-w} e^{-tf(\lambda w)}. \end{aligned} \tag{5.4.6}$$

We note that

$$\lim_{t\to\infty} \Pr\left\{L^f(t) = 0 | L^f(0) = 1\right\} = 1 \tag{5.4.7}$$

as in the classical case. From (5.4.6) we infer that the distribution of the extinction time $T_0^f = \inf\left\{t \geq 0 : L^f(t) = 0\right\}$, has the following form

$$\Pr\left\{T_0^f \in dt\right\} / dt = \int_0^\infty e^{-w} f(\lambda w) e^{-tf(\lambda w)} dw. \tag{5.4.8}$$

We now observe that all the state probabilities of the process $L(t)$ depend on the extinction probability (see [56])

$$\begin{aligned}\Pr\{L(t)=k|L(0)=1\} &= \frac{(\lambda t)^{k-1}}{(1+\lambda t)^{k+1}} && k\geq 1\\ &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left(\frac{\lambda}{1+\lambda t}\right)\\ &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left(\lambda\left(1-\Pr\{L(t)=0\}\right)\right). \end{aligned} \tag{5.4.9}$$

Hence, the state probabilities of $L^f(t)$ can be written, for $k\geq 1$, as

$$\begin{aligned}&\Pr\left\{L^f(t)=k|L^f(0)=1\right\}\\ &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\lambda\int_0^\infty\left(1-\Pr\{L(s)=0\}\right)\Pr\left\{H^f(t)\in ds\right\}\right]\\ &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\lambda\left(1-\Pr\left\{L^f(t)=0\right\}\right)\right]\\ &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{d^k}{d\lambda^k}\left[\lambda\int_0^\infty dw\, e^{-w}e^{-tf(\lambda w)}\right]. \end{aligned} \tag{5.4.10}$$

### 5.4.2 Transition probabilities

To compute the transition probabilities of $L^f(t)$, we recall that the linear birth-death process with $r$ progenitors has the following probability law (see [4], page 94, formula 8.47):

$$\Pr\{L(t)=n|L(0)=r\} = \sum_{j=0}^{min(r,n)}\binom{r}{j}\binom{r+n-j-1}{r-1}\alpha^{r-j}\beta^{n-j}(1-\alpha-\beta)^j, \tag{5.4.11}$$

where $n\geq 0$ and

$$\alpha = \frac{\mu(e^{(\lambda-\mu)t}-1)}{\lambda e^{(\lambda-\mu)t}-\mu} \qquad \text{and} \qquad \beta = \frac{\lambda(e^{(\lambda-\mu)t}-1)}{\lambda e^{(\lambda-\mu)t}-\mu}. \tag{5.4.12}$$

In the case $\lambda=\mu$ we have

$$\lim_{\mu\to\lambda}\alpha = \lim_{\mu\to\lambda}\beta = \frac{\lambda t}{1+\lambda t} \tag{5.4.13}$$

so that

$$\begin{aligned}&\Pr\{L(t)=n|L(0)=r\}\\ &= \sum_{j=0}^{min(r,n)}\binom{r}{j}\binom{r+n-j-1}{r-1}\left(\frac{\lambda t}{1+\lambda t}\right)^{r+n-2j}\left(1-2\frac{\lambda t}{1+\lambda t}\right)^j\end{aligned}$$

$$= \sum_{j=0}^{min(r,n)} \sum_{k=0}^{j} \binom{r}{j} \binom{r+n-j-1}{r-1} \binom{j}{k} (-2)^k \left( \frac{\lambda t}{1+\lambda t} \right)^{r+n-2j+k}. \tag{5.4.14}$$

One can check that for $r = 1$ the last formula reduces to

$$\Pr \{L(t) = n | L(0) = 1\} = \frac{(\lambda t)^{n-1}}{(1+\lambda t)^{n+1}}. \tag{5.4.15}$$

The transition probabilities related to the subordinated process $L^f(t)$ can be written in an elegant form, as shown in the following theorem.

**Theorem 8.** *In the subordinated linear birth-death process $L^f(t)$, when $\lambda = \mu$, $n \geq 0$, $r \geq 1$, $n \neq r$, we have that*

$$\begin{aligned} &\Pr \left\{ L^f(t+t_0) = n | L^f(t_0) = r \right\} \\ &= \sum_{j=0}^{min(r,n)} \sum_{k=0}^{j} \binom{r}{j} \binom{r+n-j-1}{r-1} \binom{j}{k} 2^k \frac{(-1)^{r+n-1} \lambda^{r+n+k-2j}}{(r+n-2j+k-1)!} \\ &\quad \times \frac{d^{r+n-2j+k-1}}{d\lambda^{r+n-2j+k-1}} \left[ \frac{1}{\lambda} - \frac{1}{\lambda} \int_0^\infty dw \, e^{-w} e^{-tf(\lambda w)} \right] \end{aligned} \tag{5.4.16}$$

*Proof.* By subordination we have

$$\begin{aligned} \Pr \left\{ L^f(t) = n | L^f(0) = r \right\} &= \int_0^\infty \Pr \{L(s) = n | L(0) = r\} \Pr \left\{ H^f(t) \in ds \right\} \\ &= \sum_{j=0}^{min(r,n)} \sum_{k=0}^{j} \binom{r}{j} \binom{r+n-j-1}{r-1} \binom{j}{k} (-2)^k \\ &\quad \times \int_0^\infty \Pr \{H(t) \in ds\} \left( \frac{\lambda s}{1+\lambda s} \right)^{r+n-2j+k}. \end{aligned} \tag{5.4.17}$$

To compute the last integral, we preliminarily observe that

$$\frac{d^m}{d\lambda^m} \frac{1}{1+\lambda s} = (-1)^m m! \, s^m \frac{1}{(1+\lambda s)^{m+1}} \tag{5.4.18}$$

and consequently

$$\left( \frac{\lambda s}{1+\lambda s} \right)^m = \frac{(-1)^{m-1} s \, \lambda^m}{(m-1)!} \frac{d^{m-1}}{d\lambda^{m-1}} \frac{1}{1+\lambda s}. \tag{5.4.19}$$

So, we have

$$\begin{aligned} &\Pr \left\{ L^f(t) = n | L^f(0) = r \right\} \\ &= \sum_{j=0}^{min(r,n)} \sum_{k=0}^{j} \binom{r}{j} \binom{r+n-j-1}{r-1} \binom{j}{k} 2^k \frac{(-1)^{r+n-1} \lambda^{r+n-2j+k}}{(r+n-2j+k-1)!} \\ &\quad \times \frac{d^{r+n-2j+k-1}}{d\lambda^{r+n-2j+k-1}} \int_0^\infty \frac{s}{1+\lambda s} \Pr \left\{ H^f(t) \in ds \right\} \end{aligned} \tag{5.4.20}$$

where, by using (5.4.6), we write

$$\begin{aligned}\int_0^\infty \frac{s}{1+\lambda s}\Pr\left\{H^f(t)\in ds\right\} &= \frac{1}{\lambda}\int_0^\infty \frac{\lambda s}{1+\lambda s}\Pr\left\{H^f(t)\in ds\right\}\\ &= \frac{1}{\lambda}\left[1-\int_0^\infty dw\, e^{-w}e^{-tf(\lambda w)}\right] \end{aligned}\tag{5.4.21}$$

and the desired result immediately follows. $\square$

**Remark 10.** *For a small time interval $dt$, the quantity in square brackets in (5.4.16) can be written as*

$$\begin{aligned}&\frac{1}{\lambda}-\frac{1}{\lambda}\int_0^\infty dw\, e^{-w}(1-dt\, f(\lambda w))\\ &= dt\,\frac{1}{\lambda}\int_0^\infty dw\, e^{-w}\int_0^\infty \nu(ds)(1-e^{-\lambda ws})\\ &= dt\int_0^\infty \nu(ds)\frac{s}{1+\lambda s}\end{aligned}$$

*Then, by using (5.4.19) e (5.4.14), formula (5.4.16) reduces to*

$$\Pr\left\{L^f(t_0+dt)=n|L^f(t_0)=k\right\} = dt\int_0^\infty \nu(ds)\Pr\left\{L(s)=n|L(0)=k\right\}$$

*thus proving relation (5.1.4) for subordinated birth-death processes.*

**Remark 11.** *If $L^f(0)=1$, from (5.4.10) we have that the probability that the number of individuals does not change during a time interval of length $dt$ is*

$$\Pr\left\{L^f(dt)=1|L^f(0)=1\right\} = 1-dt\,\frac{d}{d\lambda}\left(\lambda\int_0^\infty dw\, e^{-w}f(\lambda w)\right)$$

*Thus the waiting time for the first jump, i.e.*

$$T_1 = inf\left\{t>0 : L^f(t)\neq 1\right\},$$

*has the following distribution*

$$\Pr\left\{T_1>t\right\} = e^{-t\frac{d}{d\lambda}\left(\lambda\int_0^\infty dw\, e^{-w}f(\lambda w)\right)}.\tag{5.4.22}$$

*For example, in the case $H^f(t)$ is a stable subordinator with index $\alpha\in(0,1)$, $T_1$ has an exponential distribution with parameter $\lambda^\alpha\Gamma(\alpha+2)$.*

### 5.4.3 Mean sojourn times

Let $V_k(t)$, $k\geq 1$ the total amount of time that the process $L(t)$ spends in the state $k$ up to time $t$, i.e.

$$V_k(t) = \int_0^t I_k(L(s))\, ds,\tag{5.4.23}$$

where $I_k(.)$ is the indicator function of the state $k$. The mean sojourn time up to time $t$ is given by

$$\mathbb{E}V_k(t) = \int_0^t \Pr\{L(s) = k | L(0) = 1\} \, ds. \tag{5.4.24}$$

By means of (5.4.9) we have that

$$\begin{aligned}
\mathbb{E}V_k(t) &= \int_0^t \Pr\{L(s) = k | L(0) = 1\} \, ds \\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!} \frac{d^k}{d\lambda^k} \left( \lambda \left( t - \int_0^t \Pr\{L(s) = 0\} \, ds \right) \right) \\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!} \frac{d^k}{d\lambda^k} \left( \lambda \left( t - \int_0^t \frac{\lambda s}{1+\lambda s} ds \right) \right) \\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!} \frac{d^k}{d\lambda^k} \log(1+\lambda t) \\
&= \frac{1}{\lambda k} \left( \frac{\lambda t}{1+\lambda t} \right)^k
\end{aligned} \tag{5.4.25}$$

and the mean asymptotic sojourn time is therefore given by

$$\mathbb{E}V_k(\infty) = \frac{1}{\lambda k}. \tag{5.4.26}$$

In view of (5.4.10), for the sojourn time $V_k^f(t)$ of the subordinated process $L^f(t)$ we have that

$$\begin{aligned}
\mathbb{E}V_k^f(t) &= \int_0^t \Pr\left\{L^f(s) = k | L^f(0) = 1\right\} ds \\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!} \frac{d^k}{d\lambda^k} \left[ \lambda \int_0^\infty dw \, e^{-w} \frac{1}{f(\lambda w)} \left( 1 - e^{-tf(\lambda w)} \right) \right]
\end{aligned} \tag{5.4.27}$$

and the mean asymptotic sojourn time is given by

$$\mathbb{E}V_k^f(\infty) = \frac{(-1)^{k-1}\lambda^{k-1}}{k!} \frac{d^k}{d\lambda^k} \left[ \lambda \int_0^\infty dw \, e^{-w} \frac{1}{f(\lambda w)} \right]. \tag{5.4.28}$$

It is possible to obtain an explicit expression for $\mathbb{E}V_k^f(\infty)$ in the case of a stable subordinator, when $f(x) = x^\alpha$, $\alpha \in (0,1)$, i.e.

$$\begin{aligned}
\mathbb{E}V_k^f(\infty) &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!} \frac{d^k}{d\lambda^k} \left[ \lambda \int_0^\infty dw \, e^{-w} \frac{1}{\lambda^\alpha w^\alpha} \right] \\
&= \frac{(-1)^{k-1}\lambda^{k-1}\Gamma(1-\alpha)}{k!} \frac{d^k}{d\lambda^k} \lambda^{1-\alpha} \\
&= \frac{(-1)^{k-1}\lambda^{k-1}\Gamma(1-\alpha)}{k!} (1-\alpha)(-\alpha)(-\alpha-1)\cdots(-\alpha-k+1)\lambda^{-\alpha-k+1} \\
&= \frac{\Gamma(1-\alpha)\Gamma(\alpha+k)}{k!\Gamma(\alpha)\lambda^\alpha}
\end{aligned}$$

$$= \frac{B(1-\alpha, k+\alpha)}{\Gamma(\alpha)\lambda^\alpha}, \qquad \text{for } k \geq 1. \tag{5.4.29}$$

In the case $\alpha = \frac{1}{2}$, by using the duplication formula for the Gamma function and the Stirling formula, the quantity in (5.4.29) can be estimated, for large values of $k$, in the following way:

$$\mathbb{E}V_k^f(\infty) = \frac{\Gamma(\frac{1}{2}+k)}{k!\sqrt{\lambda}} = \frac{\Gamma(\frac{1}{2})2^{1-2k}\Gamma(2k)}{k!\sqrt{\lambda}\Gamma(k)} \simeq \frac{1}{\sqrt{\lambda k}} \tag{5.4.30}$$

which is somehow related to (5.4.26). We finally note that

$$\frac{1}{(\alpha+k)\Gamma(\alpha)\lambda^\alpha} < \mathbb{E}V_k^f(\infty) < \frac{1}{(1-\alpha)\Gamma(\alpha)\lambda^\alpha}, \qquad \forall k \geq 1, \tag{5.4.31}$$

since

$$\frac{1}{(\alpha+k)} < B(1-\alpha, k+\alpha) < \frac{1}{1-\alpha}. \tag{5.4.32}$$

### 5.4.4 On the distribution of the sojourn times

Let $L_k^f(t)$ be a linear birth-death process with $k$ progenitors. We now study the distribution of the sojourn time

$$V_k(t) = \int_0^t I_k\left(L_k^f(s)\right) ds \tag{5.4.33}$$

which represents the total amount of time that the process spends in the state $k$ up to time $t$. We now define the Laplace transform

$$r_k(\mu) = \int_0^\infty e^{-\mu t} \Pr\left\{L_k^f(t) = k\right\} dt. \tag{5.4.34}$$

The hitting time

$$V_k^{-1}(t) = \inf\{w > 0 : V_k(w) > t\} \tag{5.4.35}$$

is such that

$$\begin{aligned} \mathbb{E}\int_0^\infty e^{-\mu V_k^{-1}(t)} dt &= \mathbb{E}\int_0^\infty e^{-\mu t} dV_k(t) \\ &= \mathbb{E}\int_0^\infty e^{-\mu t} I_k\left(L_k^f(t)\right) dt \\ &= r_k(\mu). \end{aligned} \tag{5.4.36}$$

By Proposition 3.17, chapter V, of [11] we have

$$\mathbb{E}e^{-\mu V_k^{-1}(t)} = e^{-t\frac{1}{r_k(\mu)}}. \tag{5.4.37}$$

Now we resort to the fact that

$$\Pr\left\{V_k(t) > x\right\} \;=\; \Pr\left\{V_k^{-1}(x) < t\right\} \tag{5.4.38}$$

and thus we can write

$$\Pr\left\{V_k(t) \in dx\right\}/dx \;=\; -\frac{\partial}{\partial x}\int_0^t \Pr\left\{V_k^{-1}(x) \in dw\right\}. \tag{5.4.39}$$

We therefore have that

$$\begin{aligned}
\frac{1}{dx}\int_0^\infty e^{-\mu t}\Pr\left\{V_k(t)\in dx\right\}dt \;&=\; -\frac{d}{dx}\int_0^\infty dt\, e^{-\mu t}\int_0^t \Pr\left\{V_k^{-1}(x)\in dw\right\} \\
&= -\frac{d}{dx}\int_0^\infty dw \int_w^\infty dt\, e^{-\mu t}\Pr\left\{V_k^{-1}(x)\in dw\right\} \\
&= -\frac{1}{\mu}\frac{d}{dx}\int_0^\infty dw\, e^{-\mu w}\Pr\left\{V_k^{-1}(x)\in dw\right\} \\
&= -\frac{1}{\mu}\frac{d}{dx}e^{-x\frac{1}{r_k(\mu)}} \\
&= \frac{1}{\mu\, r_k(\mu)}e^{-x\frac{1}{r_k(\mu)}}.
\end{aligned} \tag{5.4.40}$$

If $r_k(0) < \infty$, from (5.4.37) it emerges that $\Pr\left\{V_k^{-1}(t) < \infty\right\} < 1$; so the sample paths of $V_k(t)$ become constant after a random time with positive probability. This is related to the fact that the subordinated birth and death process extinguishes with probability one in a finite time when $\lambda = \mu$.

We finally observe that in the case $k = 1$ by (5.4.10) we have

$$r_1(\mu) \;=\; \int_0^\infty e^{-\mu t}\Pr\left\{L^f(t) = k\right\}dt \;=\; \frac{d}{d\lambda}\left[\lambda\int_0^\infty dw\, e^{-w}\frac{1}{\mu + f(\lambda w)}\right], \tag{5.4.41}$$

provided that the Fubini Theorem holds true.

# Chapter 6

# Non homogeneous subordinators

## 6.1 Introduction

In this chapter we introduce non-decreasing jump processes with independent and time non-homogeneous increments. Although they are not Lévy processes, they somehow generalize subordinators in the sense that their Laplace exponents are possibly different Bernštein functions for each time $t$.

We call our processes *non-homogeneous subordinators* and denote them by $\sigma^{\Pi}(t)$. We investigate their basic path and distributional properties with particular attention to the governing equations.

An interesting particular case of the processes studied in the present paper is given by the the so-called multistable subordinator, which has been studied in recent years and is the main source of inspiration of our study. Multistable processes provide models to study phenomena which locally look like stable Lévy motions, but where the stability index evolves in time. There are two different types of multistable processes (for a complete discussion see [42]) . The first one is the so-called field-based process (see [24] [41]), which is neither a markovian nor a pure-jump process. The second one is the multistable process with independent increments (see [25]) with Laplace exponent

$$\Pi(\lambda, t) = \int_0^t \lambda^{\alpha(s)} ds, \tag{6.1.1}$$

which can be considered as the prototype of our non-homogeneous subordinators.

## 6.2 Non-homogeneous subordinators

Our research concerns the cádlàg processes

$$\sigma^{\Pi}(t) \;=\; b(t) + \sum_{0\leq s\leq t} \mathfrak{e}(s), \qquad t\geq 0, \tag{6.2.1}$$

where $[0,\infty)\ni t\to b(t)$ is a non-negative, differentiable function such that $b(0)=0$, and $\mathfrak{e}(s)$ is a Poisson point process in $\mathbb{R}^+$ with characteristic measure $\nu(dx,dt)$. We will work throughout the whole paper under the following assumptions

A1) $\nu(ds,\cdot)$ is absolutely continuous with respect to the Lebegue measure, i.e. there exists a density such that $\nu(ds,dt)=\nu(ds,t)dt$. Furthermore the family of measures $\{\nu(ds,t)\}_{t\geq 0}$ is such that the function $t\to\nu(ds,t)$ is continuous for each $t$.

A2) for all $t\geq 0$,

$$\int_{(0,\infty)\times[0,t]} (x\wedge 1)\nu(dx,s)ds \;<\; \infty. \tag{6.2.2}$$

We call $\sigma^{\Pi}(t)$, $t>0$, a *non-homogeneous subordinator.*

Definition (6.2.1) consists in a slight generalization of the Lévy-Itô decomposition [31] which holds for non-decreasing Lévy processes (subordinators). Therefore $\sigma^{\Pi}(t)$ retains some important properties of the usual subordinators (that is the increments are independent and the sample paths are non-decreasing) but presents a fundamental difference consisting in the non-stationarity of the increments (whose distribution is here assumed to be time-dependent). Hence, the number of points of the poissonian process in any Borel set $B\subset\mathbb{R}^+\times\mathbb{R}^+$ of the form $B=B\times[s,t]$, where $B\subset(0,\infty)$, possesses a Poisson distribution with parameter

$$m(B)=\int_B \nu(dx,s)ds \;=\; \int_B\int_{[s,t]} \nu(dx,w)dw. \tag{6.2.3}$$

In particular, the expected number of jumps of size $[x,x+dx)$ occurring up to an arbitrary instant $t$ is given by

$$\phi(dx,t)=\int_0^t \nu(dx,\tau)d\tau. \tag{6.2.4}$$

In view of (6.2.2) which implies that

$$\int_0^\infty (x\wedge 1)\phi(dx,t) \;<\; \infty \qquad \forall t>0, \tag{6.2.5}$$

we can apply Campbell theorem (see, for example, [37, p. 28]) to the process (6.2.1) in order to write that

$$\mathbb{E}e^{-\lambda\sigma^{\Pi}(t)} = e^{-\Pi(\lambda,t)} \tag{6.2.6}$$

where

$$\Pi(\lambda, t) = \lambda b(t) + \int_0^\infty \left(1 - e^{-\lambda x}\right) \phi(dx, t). \tag{6.2.7}$$

Thus the function

$$\lambda \to \Pi(\lambda, t) = \lambda b(t) + \int_0^\infty \left(1 - e^{-\lambda x}\right) \phi(dx, t) \tag{6.2.8}$$

is a Bernštein function for each value of $t \geq 0$. We recall that a Bernštein function $f$ is defined to be of class $C^\infty$ with $(-1)^{n-1} f^{(n)}(x) \geq 0$, for all $n \in \mathbb{N}$ [65, Definition 3.1]. Furthermore, a function $f$ is a Bernštein function if and only if [65, Theorem 3.2]

$$f(\lambda) = a + b\lambda + \int_0^\infty \left(1 - e^{-\lambda s}\right) \nu(ds) \tag{6.2.9}$$

where $a, b \geq 0$ and $\nu(ds)$ is a measure on $(0, \infty)$ such that

$$\int_0^\infty (s \wedge 1)\nu(ds) < \infty. \tag{6.2.10}$$

Note that under A1) and A2), and the further assumption

$$\int_0^\infty (x \wedge 1)\nu(dx, t) < \infty, \qquad \forall t \geq 0, \tag{6.2.11}$$

there exists a Bernštein function $\lambda \to f(\lambda, t)$ such that (6.2.8) can be written as

$$\Pi(\lambda, t) = \int_0^t \left( \lambda b'(w) + \int_0^\infty \left(1 - e^{-\lambda s}\right) \nu(ds, w) \right) dw = \int_0^t f(\lambda, w) dw. \tag{6.2.12}$$

In what follows the function $s \to \bar{\nu}(s, t)$ will denote the tail of the measure $\nu(ds, t)$, i.e.

$$\gamma \to \bar{\nu}(\gamma, t) = \nu((\gamma, \infty), t), \qquad \gamma > 0. \tag{6.2.13}$$

### 6.2.1 Paths properties

The process $\sigma^{\Pi}(t)$, $t \geq 0$, is the sum over a Poisson process, hence it has independent increments. As shown in the following theorems, $\sigma^{\Pi}(t)$ is continuous a.s. and, under suitable conditions, strictly increasing on any finite interval.

**Theorem 9.** *The process $\sigma^{\Pi}(t)$ is a.s. continuous, i.e., $\sigma^{\Pi}(t) = \sigma^{\Pi}(t-)$, a.s., for each fixed $t > 0$.*

*Proof.* Observe that for $h > 0$

$$\begin{aligned}
\Pr\left\{|\sigma^{\Pi}(t) - \sigma^{\Pi}(t-h)| > \epsilon\right\} &= \Pr\left\{\sum_{t-h<s\leq t} e(s) > \epsilon\right\} \\
&\leq \Pr\left\{\sum_{t-h<s\leq t} e(s)\mathbb{1}_{\{e(s)<1\}} > \frac{\epsilon}{2}\right\} \\
&\quad + \Pr\left\{\sum_{t-h<s\leq t} e(s)\mathbb{1}_{\{e(s)\geq 1\}} > \frac{\epsilon}{2}\right\}.
\end{aligned} \tag{6.2.14}$$

Now since

$$\begin{aligned}
\Pr\left\{\sum_{t-h<s\leq t} e(s)\mathbb{1}_{\{e(s)<1\}} > \frac{\epsilon}{2}\right\} &\leq \frac{2}{\epsilon}\mathbb{E}\sum_{t-h<s\leq t} e(s)\mathbb{1}_{\{e(s)<1\}} \\
&= \frac{2}{\epsilon}\int_{(t-h,t]}\int_0^1 x\nu(dx,w)dw \overset{h\to 0}{\longrightarrow} 0.
\end{aligned} \tag{6.2.15}$$

For the second term of (6.2.14) we have that

$$\begin{aligned}
\Pr\left\{\sum_{t-h<s\leq t} e(s)\mathbb{1}_{\{e(s)\geq 1\}} > \frac{\epsilon}{2}\right\} &\leq 1 - e^{-\int_1^\infty \int_{(t-h,t]} \nu(dx,w)dw} \\
&= 1 - e^{-h\nu((1,\infty),w^\star)} \\
&\overset{h\to 0}{\longrightarrow} 0.
\end{aligned} \tag{6.2.16}$$

Continuity in probability implies that for any sequence $t_n \uparrow t$ it is true that there exists a subsequence such that $\sigma^{\Pi}(t_n) \to \sigma^{\Pi}(t)$ a.s.. But since the processes $\sigma^{\Pi}(t)$ are cádlàg the left limit must exist a.s. and therefore it must be equal to $\sigma^{\Pi}(t)$. Thus the theorem is proved. □

It is well-known that a Lévy process is strictly increasing on any finite interval if the Lévy measure is supported on $(0, \infty)$ (and hence is a subordinator) and has infinite mass, $\nu(0, \infty) = \infty$ (see, for example, [64, Theorem 21.3]). In our case a similar result is true.

**Proposition 3.** *Let $W$ be a finite interval of $[0, \infty)$. If $b'(t) > 0$ for $t \in W \subseteq [0, \infty)$ the process $\sigma^{\Pi}(t)$ is a.s. strictly increasing in $W$. If $b'(t) = 0$ for $t \in W$ but $\nu((0, \infty), t) = \infty$ for all $t \in W$, then the process $\sigma^{\Pi}(t)$ is a.s. strictly increasing on $W$.*

*Proof.* If $b'(w) > 0$ for $w \in [s,t]$ then it is clear that the process $\sigma^{\Pi}(t)$ is strictly increasing. Now let $b'(t) = 0$. Observe that the Poisson point process $(\mathfrak{e}(s), s \geq 0)$ is defined as the only point $\mathfrak{e}(s) \in (0,\infty)$ such that for a Poisson random measure $\varphi(\cdot)$ on $(0,\infty) \times [0,\infty)$ with intensity $\nu(dx,dt)$ it is true that

$$\varphi|_{(0,\infty)\times\{t\}}(dx) \;=\; \delta_{(\mathfrak{e}(t),t)}(dx) \tag{6.2.17}$$

where $x \in (0,\infty) \times [0,\infty)$. Fix an interval of time $[s,t]$, then the probability that the process $\sigma^{\Pi}(t)$, $t \geq 0$, does not increase in $[s,t]$ is the probability that there are no points in $(0,\infty) \times [s,t]$. Let $E_j$, $j \in \mathbb{N}$, be a partition of $(0,\infty)$ with $\nu(E_j) < \infty$ for all $j \in \mathbb{N}$, then

$$\varphi((0,\infty) \times [s,t]) \;=\; \sum_j \varphi_j\left(((0,\infty) \cap E_j) \times [s,t]\right) \tag{6.2.18}$$

and by the countable additivity Theorem [37, p. 5] the sum (6.2.18) diverges with probability one since $\sum_j \nu(E_j) = \nu(0,\infty) = \infty$. Since this is true for any interval $[s,t]$ and if $\nu(0,\infty) = \infty$ the theorem is proved. □

### 6.2.2 Time-inhomogeneous random sums

Since the compound Poisson process is the fundamental ingredient for the costruction of a standard subordinator, it is easy to imagine that random sums with time-dependent jumps play the same role in the definition of non-homogeneous subordinators.

Let $N(t)$, $t \geq 0$, be a non-homogeneous Poisson process with intensity $g(t)$, $t \geq 0$, and let $T_j = \inf\{t > 0 : N(t) = j\}$. We consider the random sum

$$Z(t) = \sum_{j=1}^{N(t)} X(T_j) \tag{6.2.19}$$

where $X(T_j)$ is the positive-valued jump occurring at time $T_j$, having the conditional absolutely continuous distribution

$$\Pr\{X(T_j) \in dx | \, T_j = t\} = \psi(dx,t), \qquad x \geq 0, \tag{6.2.20}$$

with

$$\int_0^\infty \psi(dx,t) = 1, \qquad \forall t > 0. \tag{6.2.21}$$

The random variable $Z(t)$ takes the value $z = 0$ with positive probability, and has a density for $z > 0$. Indeed

$$\Pr\{Z(t) = 0\} = \Pr\{N(t) = 0\} = e^{-\int_0^t g(\tau)d\tau} \tag{6.2.22}$$

and, for each $z > 0$ we have that

$$\begin{aligned}
&\Pr\{Z(t) \in dz\} \\
=&\sum_{n=1}^{\infty} \int_0^t \int_{t_1}^t \dots \int_{t_{n-1}}^t \Pr\{Z(t) \in dz, T_1 \in dt_1, T_2 \in dt_2, \dots T_n \in dt_n, N(t) = n\} \\
=&\sum_{n=1}^{\infty} \int_0^t \int_{t_1}^t \dots \int_{t_{n-1}}^t \Pr\left\{ \sum_{j=1}^{N(t)} X(T_j) \in dz | \, T_1 = t_1, T_2 = t_2, \dots T_n = t_n, N(t) = n \right\} \\
&\times \Pr\left\{T_1 \in dt_1, T_2 \in dt_2, \dots T_n \in dt_n, N(t) = n\right\} \\
=&\sum_{n=1}^{\infty} \int_0^t \int_{t_1}^t \dots \int_{t_{n-1}}^t \Pr\left\{ \sum_{j=1}^{n} X(t_j) \in dz \right\} \Pr\left\{T_1 \in dt_1, \dots, T_n \in dt_n, N(t) = n\right\} \\
=&\sum_{n=1}^{\infty} \frac{1}{n!} \int_{[0,t]^n} \Pr\left\{ \sum_{j=1}^{n} X(t_j) \in dz \right\} \Pr\left\{T_1 \in dt_1, \dots T_n \in dt_n, N(t) = n\right\}. \quad (6.2.23)
\end{aligned}$$

The first factor is given by the convolution integral

$$\Pr\left\{ \sum_{j=1}^{n} X(t_j) \in dz \right\} = dz \int_{(0,\infty)^n} \psi(dx_1, t_1) \cdots \psi(dx_n, t_n) \delta\left(z - \sum_{j=1}^{n} x_j\right) \quad (6.2.24)$$

while the second one can be computed as follows

$$\Pr\left\{T_1 \in dt_1, \dots T_n \in dt_n, N(t) = n\right\} = g(t_1) \dots g(t_n) e^{-\int_0^t g(\tau) d\tau} dt_1 \dots dt_n. \quad (6.2.25)$$

Then we have

$$\begin{aligned}
&\Pr\{Z(t) \in dz\} \\
=& \, dz \sum_{n=1}^{\infty} \frac{1}{n!} \int_{[0,t]^n} dt_1 \dots dt_n \int_{(0,\infty)^n} \psi(dx_1, t_1) \cdots \psi(dx_n, t_n) \delta\left(z - \sum_{j=1}^{n} x_j\right) \\
&\times e^{-\int_0^t g(\tau) d\tau} g(t_1) \dots g(t_n). \quad (6.2.26)
\end{aligned}$$

We define the function

$$\phi(x, t) = \int_0^t g(\tau) \psi(x, \tau) d\tau. \quad (6.2.27)$$

The density of $Z(t)$ can be written as

$$\Pr\{Z(t) \in dz\} = dz \, e^{-\int_0^t g(\tau) d\tau} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(0,\infty)^n} \delta\left(z - \sum_{j=1}^{n} x_j\right) \phi(dx_1, t) \dots \phi(dx_n, t) \quad (6.2.28)$$

and therefore its Laplace transform is given by

$$\mathbb{E}e^{-\lambda Z(t)} = \Pr\{Z(t)=0\} + \int_{(0,\infty)} e^{-\lambda z} \Pr\{Z(t)\in dz\} \tag{6.2.29}$$

where

$$\begin{aligned}
&\int_{(0,\infty)} e^{-\lambda z} \Pr\{Z(t)\in dz\} \\
&= e^{-\int_0^t g(\tau)d\tau} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(0,\infty)^n} e^{-\lambda \sum_{j=1}^n x_j} \phi(dx_1,t)\dots\phi(dx_n,t) \\
&= e^{-\int_0^t g(\tau)d\tau} \sum_{n=1}^{\infty} \frac{1}{n!} \left( \int_{(0,\infty)} e^{-\lambda x}\phi(dx,t)\right)^n \\
&= e^{-\int_0^t g(\tau)d\tau} \left( e^{\int_{(0,\infty)} e^{-\lambda x}\phi(dx,t)} - 1\right).
\end{aligned} \tag{6.2.30}$$

Combining all pieces together we have that

$$\begin{aligned}
\mathbb{E}e^{-\lambda Z(t)} &= e^{-\int_0^t g(\tau)d\tau + \int_{(0,\infty)} e^{-\lambda x}\phi(dx,t)} \\
&= e^{-\int_0^t g(\tau)d\tau \int_{(0,\infty)} \psi(dx,\tau) + \int_{(0,\infty)} e^{-\lambda x} \int_0^t g(\tau)\psi(dx,\tau)d\tau} \\
&= e^{-\int_{(0,\infty)} (1-e^{-\lambda x})\phi(dx,t)}
\end{aligned} \tag{6.2.31}$$

with $\phi(x,t)$ of the form (6.2.27). Observe therefore that in this case we have that

$$\nu(dx,t) \;=\; \psi(dx,t)g(t) \text{ and } \nu((0,\infty),t) \;=\; g(t) \;<\; \infty. \tag{6.2.32}$$

### 6.2.3 Distributional properties

We remark that for each $t>0$ the distribution $\mu_t^\Pi(\bullet)$ of $\sigma^\Pi(t)$ is infinitely divisible. In fact for each $n\in\mathbb{N}$, $\mu_t^\Pi(.)$ is given by the $n$-convolution of the probability measure of the r.v.'s associated to the Lévy exponent

$$\lambda \to \frac{b(t)}{n}\lambda + \int_0^\infty \left(1-e^{-\lambda s}\right) \frac{\nu(ds,t)}{n} \tag{6.2.33}$$

where $t$ is fixed. However, unlike what happens for Lévy processes, such a distribution is not given by the $n$-th convolution of $\mu_{\frac{t}{n}}(\bullet)$ because the increments are not stationary.

It is crucial to observe that $\sigma^\Pi(t)$ can be approximated by means of a random sum of the form (6.2.19), as stated in the following theorem.

**Theorem 10.** *Let $\sigma^\Pi(t)$ be a non-homogeneous subordinator having Laplace exponent*

$$\Pi(\lambda,t) \;=\; \lambda b(t) + \int_0^\infty (1-e^{-\lambda x})\phi(dx,t) \tag{6.2.34}$$

*as in* (6.2.8) *and assume that* $s \to \bar{\nu}(s,t)$ *is absolutely continuous on* $(0,\infty)$ *for all* $t \geq 0$. *Then there exists a process* $Z_\gamma(t)$ *of type (6.2.19) such that, for* $\gamma \to 0$, *we have*

$$b(t) + Z_\gamma(t) \xrightarrow{d} \sigma^{\Pi}(t). \tag{6.2.35}$$

*Proof.* Let $\bar{\nu}$ be the function in (6.2.13). Then

$$\psi_\gamma(dx,t) \; := \; \frac{\nu(dx,t)}{\bar{\nu}(\gamma,t)} \mathbb{1}_{(\gamma,\infty)}(x) \tag{6.2.36}$$

is a probability distribution, because it is positive and integrates to 1. Let us consider the process

$$Z_\gamma(t) = \sum_{j=1}^{N_\gamma(t)} X(T_j) \tag{6.2.37}$$

where $N_\gamma(t)$ is a non-homogeneous Poisson process with rate $g_\gamma(t)$. We assume $g_\gamma(t) = \bar{\nu}(\gamma,t)$ and use (6.2.36) to write

$$\Pr\{X(T_j) \in dx | T_j = t\} = \psi_\gamma(dx,t). \tag{6.2.38}$$

In view of the discussion of Section 2.2 we have that

$$p_{Z_\gamma}(dx,t) \; = \; e^{-\int_0^t \bar{\nu}(\gamma,\tau)d\tau} \sum_{n=1}^{\infty} \frac{1}{n!} \phi^{*n}(dx,t) \mathbb{1}_{\{x>\gamma\}} + e^{-\int_0^t \bar{\nu}(\gamma,\tau)d\tau} \delta_0(dx) \tag{6.2.39}$$

and

$$\mathbb{E}e^{-\lambda b(t) - \lambda Z_\gamma(t)} \; = \; e^{-\lambda b(t) - \int_\gamma^\infty (1-e^{-\lambda x}) \int_0^t \psi_\gamma(dx,\tau) g_\gamma(\tau) d\tau} \tag{6.2.40}$$

which converges to $\mathbb{E}e^{-\lambda\sigma^{\Pi}(t)}$ as $\gamma \to 0$. $\square$

Theorem 10 provides a method to construct an approximating process. As an example, let's apply such a method to the multistable subordinator defined in Molchanov and Ralchenko [49]. In this case, the time-dependent Lévy measure is

$$\phi(dx,t) = dx \int_0^t \frac{\alpha(\tau) x^{-\alpha(\tau)-1}}{\Gamma(1-\alpha(\tau))} d\tau \tag{6.2.41}$$

for a suitable stability index $\tau \to \alpha(\tau)$ with values in $(0,1)$ in such a way that the conditions A1) and A2) are fulfilled.

The Laplace transform thus reads

$$\mathbb{E}e^{-\lambda\sigma^{\Pi}(t)} = e^{-\int_0^\infty \left(1-e^{-\lambda x}\right)\phi(dx,t)} = e^{-\int_0^t \lambda^{\alpha(s)} ds} \tag{6.2.42}$$

Being

$$\bar{\nu}(\gamma,t)=\int_\gamma^\infty \nu(dx,t)=\frac{\gamma^{-\alpha(t)}}{\Gamma(1-\alpha(t))} \tag{6.2.43}$$

the approximating process $Z_\gamma(t)$ is based on a non-homogeneous Poisson process with intensity $g_\gamma(t)=\frac{\gamma^{-\alpha(t)}}{\Gamma(1-\alpha(t))}$ and jump distribution

$$\psi_\gamma(x,t)=\gamma^{\alpha(t)}\alpha(t)x^{-\alpha(t)-1}\mathbb{1}_{[\gamma,\infty]}(x). \tag{6.2.44}$$

A convenient way to deal with the non-homogeneity of the multistable process is to consider its localizability. We remind that $\sigma^\Pi(t)$ il localizable at $t$ if the following limit holds in distribution (see, for example, [42])):

$$\lim_{r\to 0}\frac{\sigma^\Pi(t+rT)-\sigma^\Pi(t)}{r^{h(t)}}=Z_t(T) \tag{6.2.45}$$

where $Z_t(T)$, $T>0$ is the so-called local process (or tangent process) at time $t$. A fundamental property of $Z_t(T)$ is $h(t)$-self-similarity, i.e. $Z_t(rT)\overset{\mathrm{d}}{=}r^{h(t)}Z_t(T)$ for $r>0$. In the case where $\sigma^\Pi(t)$ is a multistable process, the local approximation at a fixed $t>0$ is a stable subordinator with index $\alpha(t)$. By taking the Laplace tranform,

$$\begin{aligned}\lim_{r\to 0}\mathbb{E}e^{-\lambda\frac{\sigma^\Pi(t+rT)-\sigma^\Pi(t)}{r^{h(t)}}} &= \lim_{r\to 0}\exp\left\{-\int_t^{t+rT}\left(\frac{\lambda}{r^{h(t)}}\right)^{\alpha(s)}ds\right\}\\ &= \lim_{r\to 0}\exp\left\{-\frac{rT\lambda^{\alpha(t)}}{r^{h(t)\alpha(t)}}+o(r)\right\}\\ &= e^{-T\lambda^{\alpha(t)}}\end{aligned} \tag{6.2.46}$$

where the limit produces a non-trivial result by assuming that the similarity index is $h(t)=1/\alpha(t)$.

Another way to approximate $\sigma^\Pi(t)$ is now given by means of stable processes. We split the interval $[0,t]$ into $n$ sub-intervals of length $\frac{t}{n}$ and assume $\alpha_i=\alpha\left(\frac{t}{n}i\right)$. We can write

$$\mathbb{E}e^{-\lambda\sigma^\Pi(t)}=e^{-\int_0^t\lambda^{\alpha(s)}ds}=\lim_{n\to\infty}e^{-\frac{1}{n}\sum_{i=1}^n\lambda^{\alpha_i}t}=\lim_{n\to\infty}\prod_{i=1}^n e^{-\frac{t}{n}\lambda^{\alpha_i}} \tag{6.2.47}$$

and this proves that the following equality holds in distribution

$$\sigma^\Pi(t)=\lim_{n\to\infty}\sum_{i=1}^n\left[\sigma^{\alpha_i}\left(\frac{i}{n}t\right)-\sigma^{\alpha_{i-1}}\left(\frac{i-1}{n}t\right)\right]=\lim_{n\to\infty}\sum_{i=1}^n\sigma^{\alpha_i}\left(\frac{t}{n}\right) \tag{6.2.48}$$

where $\sigma^{\alpha_i}$, $1\le i\le n$, are independent stable subordinators with index $\alpha_i=\alpha\left(\frac{t}{n}i\right)$.

To conclude this section on distributional properties, we provide sufficient conditions for the absolute continuity with respect to the Lebesgue measure of the distribution of a non-homogeneous subordinator.

**Theorem 11.** *Let $\sigma^{\Pi}(t)$, $t \geq 0$ be a non-homogeneous subordinator with Laplace exponent*

$$\Pi(\lambda, t) \;=\; \lambda b(t) + \int_0^\infty \left(1 - e^{-\lambda s}\right) \phi(ds, t) \tag{6.2.49}$$

*Suppose that $s \to \bar{\nu}(s,t)$ is absolutely continuous on $(0, \infty)$ and furthermore assume $\int_0^t \nu((0,\infty), \tau) d\tau = \infty$ for each $t$. The distribution of $\sigma^{\Pi}(t)$ is absolutely continuous with respect to the Lebesgue measure.*

*Proof.* Consider the approximating process $Z_\gamma(t)$ with distribution

$$p_{Z_\gamma}(dx, t) \;=\; e^{-\int_0^t \bar{\nu}(\gamma,\tau)d\tau} \sum_{n=1}^\infty \frac{1}{n!} \phi^{*n}(dx, t) \mathbb{1}_{\{x>\gamma\}} + e^{-\int_0^t \bar{\nu}(\gamma,\tau)d\tau} \delta_0(dx) \tag{6.2.50}$$

which converges weakly to the law of $\sigma^{\Pi}(t)$ as $\gamma \to 0$ since $Z_\gamma(t) \xrightarrow{\mathrm{d}} \sigma^{\Pi}(t)$ as $\gamma \to 0$. Consider the Lebesgue decomposition of $p_{Z_\gamma}$, written as

$$p_{Z_\gamma}(dx, t) \;=\; p^{\mathrm{d}}_{Z_\gamma}(dx, t) + p^{\mathrm{s}}_{Z_\gamma}(dx, t) + p^{\mathrm{ac}}_{Z_\gamma}(dx, t). \tag{6.2.51}$$

By hypothesis we know that

$$p^{\mathrm{d}}_{Z_\gamma}(dx, t) \;=\; e^{-\int_0^t \bar{\nu}(\gamma,\tau)d\tau} \delta_0(dx) \tag{6.2.52}$$

since $s \to \bar{\nu}(s, t)$ is absolutely continuous and therefore the Lévy measure is absolutely continous with respect to the Lebesgue measure. Therefore by letting $\gamma \to 0$ we observe

$$\int_0^\infty \left( p^{\mathrm{d}}_{Z_\gamma}(dx, t) + p^{\mathrm{s}}_{Z_\gamma}(dx, t) \right) \;=\; e^{-\int_0^t \bar{\nu}(\gamma,\tau)d\tau} \tag{6.2.53}$$

which goes to zero as $\gamma \to 0$ since $\int_0^t \bar{\nu}(0, \tau) d\tau = \infty$ and the continuity of the function $\gamma \to \int_0^t \bar{\nu}(\gamma, \tau) d\tau$ follows from the continuity of $\gamma \to \bar{\nu}(\gamma, t)$ . □

### 6.2.4 The governing equations

Under the assumptions of the above theorem, we now derive the equation governing the density of a non-homogeneous subordinator. In the following, we will denote as $q(x, t)$ the density of $\sigma^{\Pi}(t)$, when it exists, i.e.

$$\Pr\left\{\sigma^{\Pi}(t) \in dx\right\} = q(x, t) dx. \tag{6.2.54}$$

**Theorem 12.** *Let $\sigma^{\Pi}(t)$, $t \geq 0$, be a non-homogeneous subordinator and let the assumptions of Theorem 11 hold. Then a Lebesgue density of $\sigma^{\Pi}(t)$ exists and solves the variable-order equation*

$$\frac{\partial}{\partial t}q(x,t) \;=\; -b'(t)\frac{\partial}{\partial x}q(x,t) - \frac{\partial}{\partial x}\int_0^x q(s,t)\bar{\nu}(x-s,t)ds, \quad x > b(t),\, t > 0, \tag{6.2.55}$$

*provided that $q(x,t)$ is differentiable with respect to $x$, subject to $q(x,0)dx = \delta_0(dx)$ for $x \geq 0$, and $q(b(t),t) = 0$, for $t > 0$.*

*Proof.* Now we consider the Laplace transform of the right-hand side of equation (6.2.55) and we get that

$$\begin{aligned}
&\mathcal{L}\left[b'(t)\frac{\partial}{\partial x}q(x,t) + \frac{\partial}{\partial x}\int_0^x q(x,t)\bar{\nu}(x-s,t)ds\right](\lambda) \\
&= \lambda b'(t)\widetilde{q}(\lambda,t) - b'(t)q(b(t),t) + \lambda\,\mathcal{L}\left[q * \bar{\nu}\right](\lambda) \\
&= \lambda b'(t)\widetilde{q}(\lambda,t) + \lambda\widetilde{q}(\lambda,t)\left(\lambda^{-1}f(\lambda,t) - b'(t)\right) - b'(t)q(b(t),t)
\end{aligned} \tag{6.2.56}$$

where we used the fact that

$$\int_0^\infty e^{-\lambda s}\bar{\nu}(s,t)ds \;=\; \frac{1}{\lambda}f(\lambda,t) - b'(t). \tag{6.2.57}$$

Therefore the solution to (6.2.55) has Laplace transform

$$\widetilde{q}(\lambda,t) \;=\; e^{-\lambda b(t) - b(t)q(b(t),t) - \int_0^t\int_0^\infty \left(1-e^{-\lambda s}\right)\nu(ds,w)dw} \tag{6.2.58}$$

which becomes, since $q(b(t),t) = 0$,

$$\widetilde{q}(\lambda,t) \;=\; e^{-\lambda b(t) - \int_0^t\int_0^\infty \left(1-e^{-\lambda s}\right)\nu(ds,w)dw} \tag{6.2.59}$$

and coincides with $\mathbb{E}e^{-\lambda\sigma^{\Pi}(t)}$. □

If $\sigma^{\Pi}(t)$ is a multistable subordinator with index $\alpha(t)$, we have

$$\bar{\nu}(x,t) = \frac{x^{-\alpha(t)}}{\Gamma(1-\alpha(t))}, \tag{6.2.60}$$

and the governing equation reads

$$\frac{\partial}{\partial t}q(x,t) = -\frac{1}{\Gamma(1-\alpha(t))}\frac{\partial}{\partial x}\int_0^x q(y,t)\frac{1}{(x-y)^{\alpha(t)}}dy. \tag{6.2.61}$$

Keeping in mind the definition of the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$

$$\frac{\partial^\alpha}{\partial x^\alpha}u(x) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int_0^x \frac{u(y)}{(x-y)^\alpha}dy, \tag{6.2.62}$$

we can write (6.2.61) as

$$\frac{\partial}{\partial t}q(x,t) \;=\; -\frac{\partial^{\alpha(t)}}{\partial x^{\alpha(t)}}q(x,t) \qquad 0<\alpha(t)<1,\; x>0, \tag{6.2.63}$$

where $\frac{\partial^{\alpha(t)}}{\partial x^{\alpha(t)}}$ is the Riemann-Liouville derivative of time-varying order $\alpha(t)$. Then, by taking inspiration from [69, Definition 2.1], we define the generalized Riemann-Liouville derivative with kernel $\overline{\nu}(x,t)$ as

$${}^{\mathcal{R}}\mathcal{D}_x(t)\, q(x,t) = \frac{\partial}{\partial x}\int_0^x q(s,t)\overline{\nu}(x-s,t)ds \tag{6.2.64}$$

where the operator ${}^{\mathcal{R}}\mathcal{D}_x\,(t)$ acts on the variable $x$ but also depends on $t$. Using this notation, we say that the density of a non-homogeneous subordinator solves the following Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}q(x,t) = -\,{}^{\mathcal{R}}\mathcal{D}_x(t)q(x,t), & t>0, \\ q(x,0)=\delta(x). \end{cases} \tag{6.2.65}$$

It is useful to define also a generalization of the Caputo fractional derivative as

$${}^{\mathcal{C}}\mathcal{D}_x(t)q(x,t) = \int_0^x \frac{\partial}{\partial s}q(s,t)\overline{\nu}(x-s,t)ds. \tag{6.2.66}$$

If $x \to q(x,t)$ is absolutely continuous on $[0,\infty)$ then ${}^{C}\mathcal{D}_x(t)$ exists a.e. and the following relationship holds

$${}^{\mathcal{R}}\mathcal{D}_x(t)q(x,t) = q(0,t)\overline{\nu}(x,t) + {}^{\mathcal{C}}\mathcal{D}_x(t)q(x,t) \tag{6.2.67}$$

whose proof can follow [69, Proposition 2.7]. Formula (6.2.67) is a generalization of the well-known classical relationship between Caputo and Riemann-Liouville derivatives [34, page 91].

## 6.3 The inverse process

In this section we consider the process

$$L^{\Pi}(t) \;=\; \inf\left\{x \geq 0 : \sigma^{\Pi}(x) > t\right\} \tag{6.3.1}$$

where $\sigma^{\Pi}(x)$ is a non-homogeneous subordinator without drift, namely $b'(x)=0$ for all $x$. We throughout assume that

$$\nu((0,\infty),t) = \infty \text{ for all } t \geq 0 \tag{6.3.2}$$

and that

$$s \to \bar{\nu}(s,t) \;=\; \nu((s,\infty),t) \text{ is an absolutely continuous function on } (0,\infty). \tag{6.3.3}$$

By using Theorem 9 and Remark 3 it is clear that the process $L^{\Pi}$ is well defined as the inverse process of $\sigma^{\Pi}(t)$. Observe that, a.s., $L^{\Pi}(\sigma^{\Pi}(t)) = t$ since $L^{\Pi}(\sigma^{\Pi}(t)) = \inf\left\{s \geq 0 : \sigma^{\Pi}(s) > \sigma^{\Pi}(t)\right\}$ and, under (6.3.2), the process $\sigma^{\Pi}(t)$ is strictly increasing on any finite time interval (Proposition 3). In the following, we denote by $x \to l(x,t)$ the Lebesgue density of $L^{\Pi}(t)$, when such a density exists. The inverse of a classical subordinator has a Lebesgue density ([44, Theorem 3.1]). We provide here an equivalent version of [44, Theorem 3.1] valid for non-homogeneous subordinators.

**Theorem 13.** *Under the assumptions (6.3.2) and (6.3.3) the process $L^{\Pi}(t)$, $t \geq 0$, has a Lebesgue density which can be written as*

$$x \to l(x,t) \;=\; \int_0^t q(s,x)\bar{\nu}(t-s,x)ds. \tag{6.3.4}$$

*Proof.* Define

$$L(z,t) \;=\; \int_0^z l(x,t)dx \tag{6.3.5}$$

and

$$R(z,t) \;=\; \Pr\left\{L^{\Pi}(t) \leq z\right\}. \tag{6.3.6}$$

We will show that $L(z,t) = R(z,t)$. By using the convolution theorem for Laplace transform we have that

$$\widetilde{L}(z,\lambda) \;=\; \frac{1}{\lambda} - \frac{1}{\lambda}e^{-\Pi(\lambda,z)}. \tag{6.3.7}$$

The use of the relationship

$$\Pr\left\{L^{\Pi}(t) > x\right\} \;=\; \Pr\left\{\sigma^{\Pi}(x) < t\right\} \tag{6.3.8}$$

leads to the Laplace transform

$$\begin{aligned}\int_0^\infty e^{-\lambda t}R(z,t)dt \;&=\; \int_0^\infty e^{-\lambda t}\left(1 - \Pr\left\{\sigma^{\Pi}(x) < t\right\}\right)dt \\ &= \frac{1}{\lambda} - \frac{1}{\lambda}e^{-\Pi(\lambda,x)}.\end{aligned} \tag{6.3.9}$$

Therefore we have proved that

$$\int_0^\infty e^{-\lambda t}L(z,t)dt \;=\; \int_0^\infty e^{-\lambda t}R(z,t)dt \tag{6.3.10}$$

and thus in any point of continuity it is true that

$$R(z,t) \;=\; L(z,t). \tag{6.3.11}$$

If we prove that $t \to R(z,t)$ and $t \to L(z,t)$ are continuous functions then we have proved the Theorem for all $t$. Note that under (6.3.2) the process $\sigma^{\Pi}(t)$ is strictly increasing on any finite interval in view of Proposition 3. Therefore the process $L^{\Pi}(t)$ is a.s. continuous and therefore it is also continuous in distribution. This implies that $t \to R(z,t)$ is continuous. Now we show that $t \to L(z,t)$ is continuous. Note that, for $h > 0$,

$$\begin{aligned}
&|l(x,t+h) - l(x,t)| \\
=& \left| \int_0^{t+h} q(s,x)\bar{\nu}(t+h-s,x)ds - \int_0^t q(s,x)\bar{\nu}(t-s,x)ds \right| \\
=& \left| \int_0^t q(s,x)\left(\bar{\nu}(t+h-s,x) - \bar{\nu}(t-s,x)\right)ds + \int_t^{t+h} q(s,x)\bar{\nu}(t+h-s,x)ds \right| \\
\le& \int_0^t q(s,x)\left|\bar{\nu}(t+h-s,x) - \bar{\nu}(t-s,x)\right|ds + \int_t^{t+h} q(s,x)\bar{\nu}(t+h-s,x)ds \\
=& \int_0^t q(s,x)\left(\bar{\nu}(t-s,x) - \bar{\nu}(t+h-s,x)\right)ds + \int_t^{t+h} q(s,x)\bar{\nu}(t+h-s,x)ds.
\end{aligned} \tag{6.3.12}$$

Since under (6.3.3) the function $s \to \bar{\nu}(s,\bullet)$ is absolutely continuous and since

$$\bar{\nu}(t-s,x) - \bar{\nu}(t-s+h,x) \;\le\; \bar{\nu}(t-s,x) \tag{6.3.13}$$

and

$$\int_0^t \bar{\nu}(s,x)ds < \infty, \tag{6.3.14}$$

the first integral in (6.3.12) goes to zero by an application of the dominated convergence theorem. The second integral is for any $\infty > z > t$ and sufficiently small $h$

$$\int_t^{t+h} q(s,x)\bar{\nu}(t+h-s,x)ds \;=\; \int_t^z q(s,x)\mathbb{1}_{(t,t+h)}(s)\bar{\nu}(t+h-s,x)ds. \tag{6.3.15}$$

Now since

$$q(s,x)\mathbb{1}_{(t,t+h)}(s)\bar{\nu}(t+h-s,x)ds \;\le\; q(s,x)\mathbb{1}_{(t,z)}(s)\bar{\nu}(t-s,x)ds \tag{6.3.16}$$

and

$$\int_t^z q(s,x)\mathbb{1}_{(t,z)}(s)\bar{\nu}(t-s,x)ds < \infty \tag{6.3.17}$$

another application of the dominated convergence theorem shows that the second integral in (6.3.12) goes to zero. For $h < 0$ the arguments are similar. This completes the proof. □

**Theorem 14.** *If $x \to \bar{\nu}(t,x)$ is differentiable and if the density $x \to l(x,t)$ is differentiable then $l(x,t)$ solves the equation*

$$\frac{\partial}{\partial x} l(x,t) = \delta(x)\overline{\nu}(t,x) - {}^{\mathcal{R}}\mathcal{D}_t(x) l(x,t) - B_{t,x} l(x,t), \qquad x \geq 0, \tag{6.3.18}$$

*in the sense of distributions, namely it solves pointwise the Cauchy problem*

$$\begin{cases} \frac{\partial}{\partial x} l(x,t) = -{}^{\mathcal{R}}\mathcal{D}_t(x) l(x,t) - B_{t,x} l(x,t) & x > 0 \\ l(0,t) \; = \; \bar{\nu}(t,0) \end{cases} \tag{6.3.19}$$

*where ${}^{\mathcal{R}}\mathcal{D}_t(x)$ is the generalized Riemann-Liouville derivative acting on $t$ (and depending on $x$), and $B_{t,x}$ is an operator acting on both $t$ and $x$ defined as*

$$B_{t,x}\, l(x,t) = \int_0^t ds \frac{\partial}{\partial x} \overline{\nu}(t-s,x) \frac{\partial}{\partial s} \int_0^x l(x',s) dx'. \tag{6.3.20}$$

*Proof.* We can adapt [39, Theorem 8.4.1] to our case. It is sufficient to derive both sides of (6.3.4) and apply

$$\frac{\partial}{\partial t} l(x,t) = -\frac{\partial}{\partial x} q(t,x), \tag{6.3.21}$$

to obtain

$$\begin{aligned} \frac{\partial}{\partial x} l(x,t) \; &= \; \int_0^t \frac{\partial}{\partial x} q(s,x) \bar{\nu}(t-s,x) ds + \int_0^t q(s,x) \frac{\partial}{\partial x} \bar{\nu}(t-s,x) ds \\ &= - \int_0^t \frac{\partial}{\partial s} l(x,s) \bar{\nu}(t-s,x) ds - \int_0^t ds \frac{\partial}{\partial x} \overline{\nu}(t-s,x) \frac{\partial}{\partial s} \int_0^x l(x',s) dx' \\ &= -{}^{\mathcal{C}}\mathcal{D}_t(x)\, l(x,t) - \int_0^t ds \frac{\partial}{\partial x} \overline{\nu}(t-s,x) \frac{\partial}{\partial s} \int_0^x l(x',s) dx' \\ &= \delta(x)\overline{\nu}(t,x) - {}^{\mathcal{R}}\mathcal{D}_t(x)\, l(x,t) - \int_0^t ds \frac{\partial}{\partial x} \overline{\nu}(t-s,x) \frac{\partial}{\partial s} \int_0^x l(x',s) dx' \end{aligned} \tag{6.3.22}$$

where in the last step we referred to (6.2.67) □

**Remark 12.** Non stationarity is here expressed by the term $B_{t,x} l(x,t)$, which vanishes in the case of the inverse of a classical subordinator, and by the fact that the kernel of ${}^{\mathcal{R}}\mathcal{D}_t(x)$ depends on both $x$ and $t$.

In the case of the inverse of a classical stable subordinator, Theorem 14 obviously leads to the well-known Cauchy problem [46, eq (5.7)]

$$\begin{cases} \frac{\partial}{\partial x} l(x,t) \; = \; -\frac{\partial^\alpha}{\partial t^\alpha} l(x,t), & x > 0, t > 0 \\ l(0,t) \; = \; \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \end{cases} \tag{6.3.23}$$

for $\alpha \in (0,1)$.

### 6.3.1 Time changed Markov processes via the inverse of non-homogeneous subordinators.

We now consider the composition of a Markov process with the inverse of a non-homogeneous subordinator. Let $X(u)$, $u > 0$ be a Markov process in $\mathbb{R}^d$ such that $X(0) = y$ a.s. and

$$\Pr\{X(u) \in dx\} = p(x, y, u)dx. \tag{6.3.24}$$

We assume that $p(x, y, u)$ is a smooth probability density satisfying the following Cauchy problem:

$$\begin{cases} \frac{\partial}{\partial u} p = S_x p, & u > 0, \\ p(x, y, 0) = \delta(x - y), \end{cases} \tag{6.3.25}$$

where $S_x$ is the adjoint of the Markovian generator acting on the variable $x$. Moreover let $L^{\Pi}(t)$, $t \geq 0$, be the inverse of a non-homogeneous subordinator, with density as in Theorem 13

$$\Pr\{L^{\Pi}(t) \in dx\} = l(x, t)dx. \tag{6.3.26}$$

By assuming that $X(t)$ and $L^{\Pi}(t)$ are independent, we study the composition $X(L^{\Pi}(t))$, having distribution

$$\Pr\{X(L^{\Pi}(t)) \in dx\} = \int_0^{\infty} \Pr\{X(u) \in dx\} \Pr\{L^{\Pi}(t) \in du\}. \tag{6.3.27}$$

Then $X(L^{\Pi}(t))$ has a smooth density, defined as

$$g(x, y, t) = \int_0^{\infty} p(x, y, u) l(u, t) du. \tag{6.3.28}$$

By using simple arguments, we now derive the governing equation for (6.3.28).

**Proposition 4.** *Under the above assumptions, the density (6.3.28), for $t \geq 0$, solves the following equation in the sense of distributions:*

$$\begin{aligned} \int_0^{\infty} \mathcal{D}_t^R(u) \; [p(x, y, u) \, l(u, t)] \, du &= \delta(y - x)\overline{\nu}(t, 0) + S_x g(x, y, t) \\ &\quad - \int_0^{\infty} p(x, y, u) B_{t,u} \, l(u, t) du. \end{aligned} \tag{6.3.29}$$

*Proof.* We have

$$\int_0^{\infty} \mathcal{D}_t^R(u) \;\; [p(x, y, u) \, l(u, t)] \, du = \int_0^{\infty} p(x, y, u) \mathcal{D}_t^R(u) \; l(u, t) du \tag{6.3.30}$$

and by using (6.3.19) and (6.3.25), which hold for positive times, we can write

$$\begin{aligned}
&\int_0^\infty \mathcal{D}_t^R(u)\ \left[p(x,y,u)\,l(u,t)\right]du \\
&= -\lim_{\epsilon\to 0}\int_\epsilon^\infty p(x,y,u)\frac{\partial}{\partial u}l(u,t)du - \lim_{\epsilon\to 0}\int_\epsilon^\infty p(x,y,u)B_{t,u}l(u,t)du \\
&= -\lim_{\epsilon\to 0}\left[p(x,y,u)l(u,t)\right]_\epsilon^\infty + \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{\partial}{\partial u}p(x,y,u)l(u,t)du \\
&\qquad - \lim_{\epsilon\to 0}\int_\epsilon^\infty p(x,y,u)B_{t,u}l(u,t)du \\
&= \delta(y-x)\overline{\nu}(t,0) + \lim_{\epsilon\to 0}\int_\epsilon^\infty S_x p(x,y,u)\,l(u,t)du - \int_0^\infty p(x,y,u)B_{t,u}l(u,t)du \\
&= \delta(y-x)\overline{\nu}(t,0) + S_x g(x,y,t) - \int_0^\infty p(x,y,u)\,B_{t,u}\,l(u,t)du \qquad (6.3.31)
\end{aligned}$$

and the proof is complete. □

**Remark 13.** In the case $X(t)$ is a Brownian motion starting from $y$ and $L^\Pi(t)$ is the inverse of a multistable subordinator with index $\alpha(x) \in (0,1)$ we have $\overline{\nu}(t,x) = \frac{t^{-\alpha(x)}}{\Gamma(1-\alpha(x))}$ and $\mathcal{D}_t^R(x) = \frac{\partial^{\alpha(x)}}{\partial t^{\alpha(x)}}$, and thus the governing equation reads

$$\begin{aligned}
\int_0^\infty \frac{\partial^{\alpha(u)}}{\partial t^{\alpha(u)}}\ \left[p(x,y,u)\,l(u,t)\right]\ du\ &= \frac{1}{2}\Delta_x g(x,y,t) + \delta(y-x)\frac{t^{-\alpha_0}}{\Gamma(1-\alpha_0)} \\
&\quad - \int_0^\infty \frac{1}{\sqrt{2\pi u}}e^{-\frac{(y-x)^2}{2u}}B_{t,u}\,l(u,t)\,du \qquad x\geq 0
\end{aligned} \tag{6.3.32}$$

where $\alpha(0) = \alpha_0$ and

$$\begin{aligned}
B_{t,u}\,l(x,t) &= \int_0^t ds\left[\frac{\partial}{\partial u}\overline{\nu}(t-s,u)\frac{\partial}{\partial s}\int_0^u l(w,s)dw\right] \\
&= \int_0^t ds\left[\frac{\partial}{\partial u}\frac{(t-s)^{-\alpha(u)}}{\Gamma(1-\alpha(u))}\frac{\partial}{\partial s}\int_0^u l(w,s)dw\right]
\end{aligned} \tag{6.3.33}$$

Note that (6.3.32) is a generalization of the well-known fractional diffusion equation to which it reduces when $u \mapsto \alpha(u)$ is constant, that is

$$\frac{\partial^\alpha}{\partial t^\alpha}g - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\delta(x-y)\ =\ \Delta_x g \tag{6.3.34}$$

**Remark 14.** Consider the case where the Markov process is a deterministic time, namely the starting point is $y=0$ and $X(t)=t$. In this case we have $S_x = -\frac{\partial}{\partial x}$ so that the governing equation becomes, for $x \geq 0$,

$$\int_0^\infty \mathcal{D}_t^R(u)\ \left[p(x,y,u)\,l(u,t)\right]du\ = \delta(x)\overline{\nu}(t,0) - \frac{\partial}{\partial x}g - \int_0^\infty \delta(x-u)B_{t,u}\,l(u,t)du \tag{6.3.35}$$

and obviously coincides with that of $L^\Pi$ since the probability density of $X(u)$ is $p(x,0,u) = \delta(x-u)$.

## 6.4 Non-homogeneous Bochner subordination

We consider in this section a generalization of the Bochner subordination. We recall here some basic facts. Let $T_t$ be a $C_0$-semigroup of operators (the reader can consult [39] for classical information on this topic) i.e. a family of linear operators on a Banach space $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ such that, for all $u \in \mathfrak{B}$,

1. $T_0 u = u$
2. $T_t T_s u = T_{t+s} u$, $s, t \geq 0$,
3. $\lim_{t\to 0} \|T_t u - u\|_{\mathfrak{B}} = 0$.

Let $(A, \mathrm{Dom}(A))$ be the generator of $T_t$, i.e. the operator such that

$$Au := \lim_{t\to 0} \frac{T_t u - u}{t} \tag{6.4.1}$$

defined on

$$\mathrm{Dom}(A) \;=\; \left\{ u \in \mathfrak{B} : \lim_{t\to 0} \frac{T_t u - u}{t} \text{ exists as strong limit} \right\} \tag{6.4.2}$$

and let $\|T_t u\|_{\mathfrak{B}} \leq \|u\|_{\mathfrak{B}}$.

Let $\mu_t(\cdot)$ be a convolution semigroup of sub-probability measures associated with a subordinator, i.e. a family of measures $\{\mu_t\}_{t\geq 0}$ satisfying

1. $\mu_t(0, \infty) \leq 1$, for all $t \geq 0$,
2. $\mu_t * \mu_s = \mu_{t+s}$,
3. $\lim_{t\to 0} \mu_t = \delta_0$ vaguely,

and such that

$$\mathcal{L}\left[\mu_t\right](\lambda) \;=\; e^{-tf(\lambda)}, \tag{6.4.3}$$

where

$$f(\lambda) \;=\; a + b\lambda + \int_0^\infty \left(1 - e^{-\lambda s}\right) \nu(ds) \tag{6.4.4}$$

is a Bernštein function. The operator defined by the Bochner integral

$$T_t^f u \;=\; \int_0^\infty T_s u \, \mu_t(ds), \qquad u \in \mathfrak{B}, \tag{6.4.5}$$

is said to be a subordinate semigroup in the sense of Bochner. A classical result due to Phillips [62] states that $T_t^f$ is again a $C_0$-semigroup and is generated by

$$-f(-A)u := -au + bAu + \int_0^\infty (T_s u - u)\,\nu(ds) \tag{6.4.6}$$

which is always defined at least on $\mathrm{Dom}(A)$ [65, Theorem 12.6].

In order to extend such a result to non-homogeneous evolutions, a generalization of the notion of one-parameter semigroup is needed. Let $(\mathfrak{B}, \|.\|_{\mathfrak{B}})$ be a Banach space. A family of mappings $\mathcal{T}_{s,t}$ from $\mathfrak{B}$ to itself, defined by the pair of numbers $s$ and $t$ (such that $0 \leq s \leq t$), is said to be a propagator (two-parameter semigroup) if for each $u \in \mathfrak{B}$, [39, Section 1.9]

1. $\mathcal{T}_{t,t}u = u$, for each $t \geq 0$;
2. $\mathcal{T}_{s,t}\mathcal{T}_{r,s}u = \mathcal{T}_{r,t}u$, for $r \leq s \leq t$;
3. $\lim_{\delta \to 0} \|\mathcal{T}_{s+\delta,t}u - \mathcal{T}_{s,t}u\|_{\mathfrak{B}} = \lim_{\delta \to 0} \|\mathcal{T}_{s,t+\delta}u - \mathcal{T}_{s,t}u\|_{\mathfrak{B}} = 0$;

It is obvious that a propagator $\mathcal{T}_{s,t}$ reduces to a classical one-parameter semigroup in the case where it only depends on the difference $t - s$.

Let $\sigma^{\Pi}(t)$, $t \geq 0$, be a non-homogeneous subordinator and consider the measures $\mu_{s,t}(.)$ corresponding to the distribution of the increments $\sigma^{\Pi}(t) - \sigma^{\Pi}(s)$ which are obviously such that

$$\mathcal{L}[\mu_{s,t}](\lambda) = e^{-\int_s^t f(\lambda,\tau)d\tau} \tag{6.4.7}$$

as can be ascertained by applying the Campbell theorem to $\sigma^{\Pi}(t) - \sigma^{\Pi}(s)$ under assumption (6.2.11). Therefore, it is easy to verify that the family of measures $\{\mu_{s,t}(\cdot)\}_{0 \leq s \leq t}$ forms a two-parameter convolution semigroup of probability measures since, from the independence of the increments and (6.4.7), we get $\mu_{s,t} * \mu_{r,s} = \mu_{r,t}$, $r \leq s \leq t$. Consider the operator defined by the Bochner integral on $\mathfrak{B}$

$$\mathcal{T}_{s,t}u = \int_0^\infty T_\omega u\, \mu_{s,t}(d\omega). \tag{6.4.8}$$

The family of operators $\{\mathcal{T}_{s,t}\}_{0 \leq s \leq t}$ forms a two-parameter semigroup of operators on $\mathfrak{B}$, i.e., (6.4.8) is a propagator. This can be easily ascertained by observing that for all $u \in \mathfrak{B}$

$$\begin{aligned}
\mathcal{T}_{s,t}\mathcal{T}_{r,s}u &= \int_0^\infty T_w \left[\int_0^\infty T_{w'} u\, \mu_{r,s}(dw')\right] \mu_{s,t}(dw) \\
&= \int_0^\infty \int_0^\infty T_{w+w'} u\, \mu_{r,s}(dw')\, \mu_{s,t}(dw)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \int_w^\infty T_\rho u \, \mu_{r,s}(d(\rho - w)) \, \mu_{s,t}(dw) \\
&= \int_0^\infty T_\rho u \int_0^\rho \mu_{s,t}(d(\rho - w)) \, \mu_{r,s}(dw) \\
&= \int_0^\infty T_\rho u \, \mu_{r,t}(d\rho) \\
&= \mathcal{T}_{r,t} u.
\end{aligned} \tag{6.4.9}
$$

We consider here the case where the generator $(A, \mathrm{Dom}(A))$ of $T_t$ is a self-adjoint, dissipative operator on an Hilbert space $(\mathfrak{H}, \langle \cdot, \cdot \rangle)$ and thus we have that $\|T_t u\|_{\mathfrak{H}} \leq \|u\|_{\mathfrak{H}}$ (see, for example, [32, Section 2.7] and [65, Chapter 11] for classical information on linear operators on Hilbert spaces). Recall that an operator is said to be dissipative if $\langle Au, u \rangle \leq 0$ for all $u \in \mathrm{Dom}(A)$ and

$$
\mathrm{Dom}(A) \; = \; \left\{ u \in \mathfrak{H} : \|Au\|_{\mathfrak{H}}^2 < \infty \right\}. \tag{6.4.10}
$$

**Theorem 15.** *Let the above assumptions (including (6.2.11)) be fulfilled. The family of operators $\mathcal{T}_{s,t}$ acting on an element $u \in \mathfrak{H}$ is a bounded propagator on $\mathfrak{H}$ and for $u \in Dom(A)$, the map $t \to \mathcal{T}_{s,t} u$ solves,*

$$
\begin{cases}
\frac{d}{dt} q(t) \; = \; -f(-A, t) q(t), & 0 \leq s \leq t, \\
q(s) = u \in Dom(A),
\end{cases} \tag{6.4.11}
$$

*where the family of generators $\{-f(-A,t)\}_{t \geq 0}$, can be defined as*

$$
-f(-A, t) q \; := \; b'(t) A q + \int_0^\infty (T_s q - q) \, \nu(ds, t), \tag{6.4.12}
$$

*a Bochner integral on $Dom(A)$.*

*Proof.* First note that

$$
\|\mathcal{T}_{s,t} u\|_{\mathfrak{H}} \; \leq \; \int_0^\infty \|T_w u\|_{\mathfrak{H}} \, \mu_{s,t}(dw) \; \leq \; \|u\|_{\mathfrak{H}} \, , \tag{6.4.13}
$$

and therefore $\mathcal{T}_{s,t}$ is bounded.

Then we recall ([65, Theorem 11.4] and [32, Theorem 2.7.30]) that within such a framework we have by the spectral Theorem that for $u \in \mathrm{Dom}(A)$

$$
Au \; = \; \int_{(-\infty, 0]} \lambda \, E(d\lambda) u \tag{6.4.14}
$$

where $E(B) : \mathrm{Dom}(A) \to \mathrm{Dom}(A)$, $B$ a Borel set of $\mathbb{R}$, is an orthogonal projection-valued measure supported on the spectrum of $A$ defined as

$$
E(B) u := \int_B E(d\lambda) u. \tag{6.4.15}
$$

Therefore since from (6.4.14) it is true that for a function $\Phi : (-\infty, 0] \to \mathbb{R}$

$$\Phi(A)u = \int_{(-\infty,0]} \Phi(\lambda)E(d\lambda)u \tag{6.4.16}$$

we have that

$$T_t u = \int_{(-\infty,0]} e^{t\lambda}E(d\lambda)u. \tag{6.4.17}$$

We now verify that $\mathcal{T}_{s,t}\mathcal{T}_{r,s}u = \mathcal{T}_{r,t}u$, $r \le s \le t$, to show that for all $u \in \mathfrak{H}$ the operator $\mathcal{T}_{s,t}$ is a propagator since the other defining properties as trivially verified. We have that, for all $u \in \mathfrak{H}$,

$$\begin{aligned}
\mathcal{T}_{s,t}\mathcal{T}_{r,s}u &= \int_0^\infty \int_0^\infty T_{w+\rho}u\, \mu_{r,s}(dw)\mu_{s,t}(d\rho) \\
&= \int_0^\infty \int_0^\infty \int_{(-\infty,0]} \int_{(-\infty,0]} e^{\lambda w}e^{\varrho\rho}E(d\lambda)E(d\varrho)u\, \mu_{r,s}(dw)\mu_{s,t}(d\rho) \\
&= \int_{(-\infty,0]} \int_{(-\infty,0]} e^{-\int_r^s f(-\lambda,w)dw}e^{-\int_s^t f(-\varrho,w)dw}E(d\lambda)E(d\varrho)u \\
&= \int_{(-\infty,0]} e^{-\int_r^t f(-\lambda,w)dw}E(d\lambda)u \\
&= \int_0^\infty \int_{(-\infty,0]} e^{\lambda w}\mu_{r,t}(dw)E(d\lambda)u \\
&= \int_0^\infty T_w u\, \mu_{r,t}(dw) \\
&= \mathcal{T}_{r,t}u.
\end{aligned} \tag{6.4.18}$$

For a function $u$ such that

$$\int_{(-\infty,0]} |f(-\lambda,t)|^2 \langle E(d\lambda)u, u\rangle < \infty \tag{6.4.19}$$

the representation (6.4.12) can be shown to be true: use (6.4.16) to write

$$\begin{aligned}
-f(-A,t)u &= -\int_{(-\infty,0]} f(-\lambda,t)E(d\lambda)u \\
&= -\int_{(-\infty,0]} \left(-b'(t)\lambda + \int_0^\infty \left(1 - e^{\lambda s}\right)\nu(ds,t)\right) E(d\lambda)u \\
&= \int_{(-\infty,0]} b'(t)\lambda E(d\lambda)u + \int_0^\infty \int_{(-\infty,0]} \left(e^{\lambda s} - 1\right) E(d\lambda)u\, \nu(ds,t) \\
&= b'(t)Au + \int_0^\infty (T_s u - u)\, \nu(ds,t).
\end{aligned} \tag{6.4.20}$$

Now we show that (6.4.20) is true for any $u \in \mathrm{Dom}(A)$

$$\|f(-A,t)u\|_{\mathfrak{H}} \le b'(t)\,\|Au\|_{\mathfrak{H}} + \int_0^\infty \|T_s u - u\|_{\mathfrak{H}}\, \nu(ds,t)$$

$$\leq b'(t) \|Au\|_{\mathfrak{H}} + \int_0^1 s \|Au\|_{\mathfrak{H}} \nu(ds,t) + 2\int_1^\infty \|u\|_{\mathfrak{H}} \nu(ds,t). \quad (6.4.21)$$

Now note that

$$\begin{aligned} \mathcal{T}_{s,t}u &= \int_0^\infty T_w u \, \mu_{s,t}(dw) \\ &= \int_0^\infty \left[ \int_{(-\infty,0]} e^{w\lambda} E(d\lambda) u \right] \mu_{s,t}(dw) \\ &= \int_{(-\infty,0]} e^{-\int_s^t f(-\lambda,\tau)d\tau} E(d\lambda) u \end{aligned} \quad (6.4.22)$$

where we used (6.4.16). The fact that $\mathcal{T}_{s,t}$ maps $\mathrm{Dom}(A)$ into itself can be ascertained by using again [65, Theorem 11.4] for saying that $E(\cdot)$ maps $\mathrm{Dom}(A)$ into itself and furthermore, since $E(I)E(J) = E(I \cap J)$ for any $I, J$ Borel sets of $\mathbb{R}$, we observe that for any $u \in \mathrm{Dom}(A)$

$$\begin{aligned} \mathcal{T}_{s,t}Au &= \int_{(-\infty,0]} e^{-\int_s^t f(-\lambda,w)dw} E(d\lambda) \int_{(-\infty,0]} \mu E(d\mu) u \\ &= \int_{(-\infty,0]} \lambda e^{-\int_s^t f(-\lambda,w)dw} E(d\lambda) u \\ &= \int_{(-\infty,0]} \mu E(d\mu) \int_{(-\infty,0]} e^{-\int_s^t f(-\lambda,w)dw} E(d\lambda) u \\ &= A\mathcal{T}_{s,t}u. \end{aligned} \quad (6.4.23)$$

Now note that the equality

$$\frac{d}{dt}\mathcal{T}_{s,t}u = -f(-A,t)\mathcal{T}_{s,t}u, \qquad 0 \leq s \leq t, \quad (6.4.24)$$

must be true in the sense of (6.4.16) and indeed by using (6.4.22) we have that, for $u \in \mathrm{Dom}(A)$,

$$\begin{aligned} \frac{d}{dt}\mathcal{T}_{s,t}u &= \int_{(-\infty,0]} \frac{d}{dt} e^{-\int_s^t f(-\lambda,w)dw} E(d\lambda) u \\ &= -\int_{(-\infty,0]} f(-\lambda,t) e^{-\int_s^t f(-\lambda,w)dw} E(d\lambda) u \\ &= -\int_{(-\infty,0]} f(-\mu,t) E(d\mu) \int_{(-\infty,0]} e^{-\int_s^t f(-\lambda,w)dw} E(d\lambda) u \\ &= -f(-A,t)\mathcal{T}_{s,t}u \end{aligned} \quad (6.4.25)$$

where we used again [65, Theorem 11.4]. □

### 6.4.1 Time-changed Brownian motion via non-homogeneous subordinators

In this section we provide some basic facts concerning Brownian motion time-changed with a non-homogeneous subordinator. This is the immediate generalization of the classical subordinate Brownian motion: the reader can consult [12; 35; 36; 66] for recent developments on this point. Therefore we assume now that

$$T_t u \;=\; \mathbb{E}^x u(B(t)), \qquad t \geq 0, \tag{6.4.26}$$

for $u \in L^2(\mathbb{R}^n)$ where $B$ is an $n$-dimensional Brownian motion starting from $x \in \mathbb{R}^n$. We have therefore the formal representation

$$T_t u \;=\; e^{\frac{1}{2}t\Delta} u \tag{6.4.27}$$

where $\Delta$ is the $n$-dimensional Laplace operator such that

$$\mathrm{Dom}(\Delta) \;=\; \left\{ u \in L^2(\mathbb{R}^n) : \|\Delta u\|_{L^2(\mathbb{R}^n)} < \infty \right\}. \tag{6.4.28}$$

Therefore we get

$$\mathcal{T}_{s,t} u \;=\; \mathbb{E}^x u\left(B\left(\sigma^\Pi(t) - \sigma^\Pi(s)\right)\right), \qquad 0 \leq s \leq t, u \in L^2\left(\mathbb{R}^n\right). \tag{6.4.29}$$

Consider, for example, the case of a multistable subordinator, where $\Pi(\lambda, t) = \int_0^t \lambda^{\alpha(s)} ds$ for a suitable choice of $\alpha(s)$ with values in $(0,1)$. Then $\lambda \to \lambda^{\alpha(s)}$ is a Bernštein function for each $s \geq 0$, and Theorem 15 leads to

$$-\left(-\Delta\right)^{\alpha(t)} u \;=\; \frac{\alpha(t)}{\Gamma(1-\alpha(t))} \int_0^\infty \left(T_s u - u\right) s^{-\alpha(t)-1} ds \tag{6.4.30}$$

for a function $u \in \mathrm{Dom}(\Delta)$. Note that in this case we have a Brownian motion composed with the multistable subordinator whose increments have characteristic function

$$\mathbb{E} e^{i\xi B\left(\sigma^\Pi(t) - \sigma^\Pi(s)\right)} \;=\; e^{-\int_s^t (\|\xi\|^2/2)^{\alpha(w)} dw}. \tag{6.4.31}$$

By following, for example, [23, Section 3.1] the generator (6.4.30) can be also defined as

$$-\left(-\Delta\right)^{\alpha(t)} u \;=\; -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \, \|\xi\|^{2\alpha(t)} \, \widehat{u}(\xi) d\xi \tag{6.4.32}$$

with

$$\mathrm{Dom}\left(\left(-\Delta\right)^{\alpha(t)}\right) \;=\; \left\{ u \in L^2\left(\mathbb{R}^n\right) : \int_{\mathbb{R}^n} \|\xi\|^{2\alpha(t)} \, \widehat{u}(\xi) d\xi < \infty, \text{ for each } t \geq 0 \right\}. \tag{6.4.33}$$

In general, we observe that for any non-homogeneous subordinator we can write

$$\mathbb{E} e^{i\xi B\left(\sigma^\Pi(t) - \sigma^\Pi(s)\right)} \;=\; e^{-\int_s^t f\left(\left\|\frac{\xi}{2}\right\|^2, w\right) dw} \tag{6.4.34}$$

and we can adapt [32, Example 4.1.30] to write

$$-f\left(-\frac{1}{2}\Delta, t\right) u = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\xi\cdot x} f\left(\left\|\frac{\xi}{2}\right\|^2, t\right)\widehat{u}(\xi)d\xi \tag{6.4.35}$$

with

$$\mathrm{Dom}\left(f\left(-\frac{1}{2}\Delta, t\right)\right) = \left\{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} f\left(\left\|\frac{\xi}{2}\right\|^2, t\right)\widehat{u}(\xi)d\xi < \infty, \forall t \geq 0\right\}. \tag{6.4.36}$$

Therefore, we have by Theorem 15 the structure of the solution to a sort of diffusion equation

$$\frac{d}{dt}q(t) = -f\left(-\frac{1}{2}\Delta, t\right) q(t). \tag{6.4.37}$$

We investigate here the mean square displacement i.e. the quantity

$$\mathcal{M}(t) = \int_{\mathbb{R}^n} \|x\|^2 \Pr\left\{B\left(\sigma^{\Pi}(t)\right) \in dx\right\}. \tag{6.4.38}$$

Roughly speaking, a stochastic process is said to have a diffusive asymptotic behaviour when $\mathcal{M}(t) \sim Ct$ i.e. the mean square displacement grows linearly with time. When $\mathcal{M}(t) \sim t^{\alpha}$, $\alpha \in (0,1)$, the process is said to be subdiffusive, while if $\alpha > 1$ it is super-diffusive (the reader can consult [47; 48] for an overview on anomalous diffusive behaviours). Here it is interesting to note that the mean value of the Lévy measure, namely $\int_0^\infty w\nu(dw,t)$ determines under which conditions the asymptotic behavior is respectively diffusive, sub-diffusive or super-diffusive.

**Proposition 5.** *We have the following behaviours.*

1. *If and only if*

$$\int_1^\infty w\phi(dw,t) < \infty \text{ for } 0 \leq t < t_0 \leq \infty \tag{6.4.39}$$

*it is true that $\mathcal{M}(t) < \infty$ for all $t < t_0$*

2. *Under (6.4.39) for $t_0 = \infty$, we have that*

$$0 < \lim_{t\to\infty}\frac{\mathcal{M}(t)}{t} = C < \infty \text{ if and only if } \lim_{t\to\infty}\int_0^\infty w\nu(dw,t) = C \tag{6.4.40}$$

$$\lim_{t\to\infty}\frac{\mathcal{M}(t)}{t} = \infty \text{ if and only if } \lim_{t\to\infty}\int_0^\infty w\nu(dw,t) = \infty \tag{6.4.41}$$

3. *Under* (6.4.39) *for* $t_0 = \infty$, *if*

$$\lim_{t\to\infty} \int_0^\infty w\nu(dw, t) = 0 \tag{6.4.42}$$

*then*

$$\lim_{t\to\infty} \frac{\mathcal{M}(t)}{t} = 0. \tag{6.4.43}$$

*Proof.* Observe that under (6.4.39)

$$\begin{aligned}
\mathcal{M}(t) &= \int_{\mathbb{R}^n} \|x\|^2 \int_0^\infty \Pr\{B(s) \in dx\} \Pr\left\{\sigma^\Pi(t) \in ds\right\} \\
&= n \int_0^\infty s \Pr\left\{\sigma^\Pi(t) \in ds\right\} \\
&= -n \frac{d}{d\lambda} e^{-\Pi(\lambda,t)}\Big|_{\lambda=0} \\
&= n \int_0^\infty w\phi(dw, t) < \infty \text{ if } t < t_0.
\end{aligned} \tag{6.4.44}$$

Observe that the last integral in (6.4.44) converges only under (6.4.39). Now note that

$$\begin{aligned}
\lim_{t\to\infty} \frac{\mathcal{M}(t)}{t} &= \frac{n \int_0^\infty w\phi(dw, t)}{t} \\
&= \lim_{t\to\infty} \frac{n \int_0^t \int_0^\infty w\nu(dw, s)ds}{t}
\end{aligned} \tag{6.4.45}$$

and therefore the proof of Item (2) and (3) is easy to be done. □

A time-change by means of a multistable subordinator leads in this case to a process with $\mathcal{M}(t) = \infty$ for any $t$ as a consequence of Item 1 of Proposition 5. Consider now the measure

$$\nu(ds, t) = s^{-1} e^{-\alpha(t)s}\, ds \tag{6.4.46}$$

for a function $\alpha(t) > 0$ such that A1) and A2) are fulfilled. The associated Bernštein functions become, for each $t \geq 0$,

$$f(\lambda, t) = \log\left(1 + \frac{\lambda}{\alpha(t)}\right) \tag{6.4.47}$$

and in view (6.4.46) we can compute

$$\mathcal{M}(t) = n \int_0^t \frac{d\tau}{\alpha(\tau)}. \tag{6.4.48}$$

Observe that Proposition 5 leads to the study of the limit

$$\lim_{t\to\infty}\int_0^\infty e^{-\alpha(t)w}dw \;=\; \lim_{t\to\infty}\frac{1}{\alpha(t)} \tag{6.4.49}$$

therefore the asymptotic behaviour of $\mathcal{M}(t)$ in this case depends on the asymptotic behaviour of $\alpha(t)$.

If instead, for functions $\alpha(t)$ strictly between zero and one and $\theta(t) > 0$ as in A1) and A2),

$$\nu(ds,t) \;=\; \frac{\alpha(t)s^{-\alpha(t)-1}\,e^{-\theta(t)s}}{\Gamma(1-\alpha(t))}ds \tag{6.4.50}$$

then the Bernštein functions are a generalization of the Laplace exponent of the relativistic stable subordinator

$$f(\lambda,t) \;=\; (\lambda+\theta(t))^{\alpha(t)} - \theta(t)^{\alpha(t)} \tag{6.4.51}$$

and the asymptotic behaviour of the $\mathcal{M}(t)$ is determined in this case by the limit

$$\lim_{t\to\infty}\int_0^\infty \frac{\alpha(t)s^{-\alpha(t)}\,e^{-\theta(t)s}}{\Gamma(1-\alpha(t))}ds \;=\; \lim_{t\to\infty}\alpha(t)\,\theta(t)^{\alpha(t)-1}. \tag{6.4.52}$$

The explicit form of $\mathcal{M}(t)$ is here

$$\mathcal{M}(t) \;=\; n\int_0^t \frac{\alpha(\tau)}{\theta(\tau)^{1-\alpha(\tau)}}d\tau. \tag{6.4.53}$$

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